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OPTIMAL REPLACEMENT UNDER ADDITIVE DAMAGE AND OTHER FAILURE MODELS

Howard M. Taylor*

Center for Environmental Quality Management
Cornell University

ABSTRACT

A machine or production system is subject to random failure. Upon failure the system is replaced by a new one, and the process repeats. A cost is associated with each replacement, and an additional cost is incurred at each failure in service. Thus, there is an incentive for a controller to attempt to replace before failure occurs. The problem is to find an optimal control strategy that balances the cost of replacement with the cost of failure and results in a minimum total long-run average cost per unit time. We attack this problem under the cumulative damage model for system failure. In this failure model, shocks occur to the system in accordance with a Poisson process. Each shock causes a random amount of damage or wear and these damages accumulate additively. At any given shock, the system fails with a known probability that depends on the total damage accumulated to date. We assume that the cumulative damage is observable by the controller and that his decisions may be based on its current value. Supposing that the shock failure probability is an increasing function of the cumulative damage, we show that an optimal policy is to replace either upon failure or when this damage first exceeds a critical control level, and we give an equation which implicitly defines the optimal control level in terms of the cost and other system parameters. Also treated are some more general models that allow for income lost during repair time and other extensions.

1. INTRODUCTION AND SUMMARY

Consider a machine or production system which generates income while operating, but which is subject to random failure. Upon failure the system is replaced with a new one, or repaired to a like new condition, and the process repeats. Each replacement costs c dollars and each failure adds a cost of K dollars, thus providing an incentive for a controller to attempt to replace before failure occurs. The problem is to find an optimal control strategy that balances the cost of replacement with the cost of failure and results in a minimum total long run average cost per unit time.

Such failure versus planned replacement problems have been considered by a number of researchers. Barlow and Proschan [2], chap. 3, and Cox [7], chap. 11, review much of the early work. For the most part, these researchers assumed that the age of the system in current service was maintained as a control variable, but that no further information on the state of the production system was available. To replace either upon failure or upon first reaching a fixed critical age level, whichever occurs first, was shown to be an optimal policy, and a means of determining the optimal critical age level was given.

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We attack a similar problem in this article, but under the *cumulative damage model* for system failure. In this failure model, shocks occur to the system in accordance with a Poisson process. Each shock causes a random amount of damage and these damages accumulate additively. At any given shock, the system fails with a known probability that depends on the total damage accumulated to date. Cherniavsky [6] discusses these and other shock failure models in considerable detail. A predecessor model, in which the system fails when the total damage first exceeds a fixed threshold, is reviewed by Buckland [4], sect. 1.10.

We suppose that the accumulated damage, possibly representing wear (Mercer [13]), is observable by the controller, and thus his decisions may be based on its current value. Assuming that the shock failure probability is an increasing function of the cumulative damage, we show that an optimal policy is to replace either upon failure or when the accumulated damage first exceeds a critical control level ξ^*, and we give an equation which implicitly defines ξ^* in terms of the cost and other system parameters.

Section 2 describes our failure model in detail, and records some of its properties for later reference. Section 3 solves the replacement problem just mentioned, and section 4 treats the special case in which the system fails when the cumulative damage first exceeds a threshold. Especially simple results are obtained here when the damages are exponentially distributed. Section 5 generalizes the basic model to include income lost during repair time.

A failure model in which system failure is determined by the number of shocks, and not their magnitudes, is treated next. The next two sections discuss related optimal stopping problems. The article closes with a numerical example showing how lower system operating costs result from increased control information.

2. THE FAILURE MODEL

In this section we describe the failure model in detail and record some of its properties for later reference.

Shocks occur to the system in accordance with a Poisson process having rate λ. Each shock causes a random amount of damage, and these damages accumulate additively, up to the moment of failure. Roughly speaking, if at time t the cumulative damage is $X(t)=x$, and a shock of magnitude y occurs, thus increasing the cumulative damage to $x+y$, then the system fails with known probability $1-r(x+y)$. In the event failure does not occur, after an exponentially distributed wait, another shock happens, and the process repeats. The successive shock magnitudes $Y_1, Y_2, \ldots$ are independent identically distributed positive random variables having the known distribution function $F(y)=Pr\{Y_k \leq y\}$.

Let ζ denote the time of system failure, and for $t<\zeta$, let $X(t)$ denote the cumulative damage attributed to shocks occurring during $[0, t]$. This damage process $\{X(t);\ 0 \leqq t < \zeta\}$ is a (terminating) Markov process, and we may precisely specify it by translating our earlier rough description into a recipe for its infinitesimal operator G. While in our application the system always begins without damage, that is $X(0)=0$, to fully exploit the Markov property, it behooves us to consider other initial values. Let E_x denote expectation conditional on $X(0)=x$ (and reserve E without affixes for expectation under our reference probability measure, where $X(0)=0$). The domain of the infinitesimal operator G is the set of bounded Borel measurable functions ϕ defined on $[0, \infty)$. Motivated by our earlier rough description, for such ϕ we prescribe

$$(2.1)\qquad G\phi(x)=\lim t^{-1}\{E_x[\phi(X(t));t<\zeta]-\phi(x)\}$$

$$=\lim_{t\downarrow 0} t^{-1}\left\{(1-\lambda t)\phi(x)+\lambda t\int r(x+y)\phi(x+y)dF(y)+o(t)-\phi(x)\right\}$$

$$=-\lambda\left[\phi(x)-\int r(x+y)\phi(x+y)dF(y)\right],$$

where the limit is taken boundedly pointwise, or even uniformly, in x. Notation: If A is an event, we set $E_x[\cdot;A]=\int_A \cdot\, dP_x$, where P_x is the probability on sample paths corresponding to E_x.) Chapter 15 section 4 of Breiman [5] provides a helpful discussion of infinitesimal operators of nonterminating, pure jump Markov processes.

A *Markov time* in the context of a terminating process is a random variable T satisfying $0\leqq T\leqq\zeta$ and for which, for all t, the event $\{T\geqq t\}$ is in the appropriate completion of the sigma algebra generated by $\{X(s);\,0\leqq s\leqq t\}$ in the space $\{\zeta\geqq t\}$. Of great importance to us is *Dynkin's formula*

$$(2.2)\qquad E_x[\phi(X(T));T<\zeta]-\phi(x)=E_x\left[\int_0^T G\phi(X(s))ds\right],$$

valid for any ϕ in the domain of G and any Markov time T having finite expectation (Theorem 5.1 and its corollary in Dynkin [11]).

Let N be the index of the shock at which failure occurs, or equivalently, the number of shocks until failure. Expressing ζ as the sum of a random number of exponentially distributed intershock times, each having mean λ^{-1}, enables us to derive rather easily a formula for

$$(2.3)\qquad w(x)=E_x[\zeta].$$

We introduce the substochastic kernel

$$(2.4)\qquad K(x,A)=E[r(x+Y_1);\,x+Y_1\epsilon A]$$

$$=Pr\{N>1\quad\text{and } x+Y_1\epsilon A\},$$

for $x\epsilon[0,\infty)$ and A a Borel subset of $[0,\infty)$. The kernel is determined by the survival probability r and damage distribution function F through

$$K(x,[0,z])=\int_0^{(z-x)^+} r(x+y)dF(y).$$

Let K^0 be the identity kernel, wherein $K^0(x,A)=1$ if $x\epsilon A$ and 0, otherwise, and define

$$K^n(x, A) = \int K(x, dz)K^{n-1}(z, A) \tag{2.5}$$

$$= Pr\{N > n \quad \text{and } x + Y_1 + \ldots + Y_n \in A\}.$$

Finally set

$$H(x, A) = \sum_{n=0}^{\infty} K^n(x, A).$$

Of course, $H(x, A)$ gives the mean number of n's satisfying both $N > n$ and $x + Y_1 + \ldots + Y_n \in A$. Since each $n < N$ has an associated intershock time of mean λ^{-1}, we obtain

$$w(x) = \lambda^{-1} H(x, [0, \infty)). \tag{2.6}$$

This formula holds whether or not $w(x)$ is finite.

The reader familiar with renewal processes might benefit from observing the similarity between $H(0, [0, z])$ and the renewal function $M(z) = \sum_{n \geqq 0} \Pr\{Y_1 + \ldots + Y_n \leqq z\}$, which records the mean number of n's for which $Y_1 + \ldots + Y_n \leqq z$. Of course $H(0, [0, z]) \leqq M(z)$ since the former is diminished by the possibility of failure. This similarity with renewal theory may help motivate several of the formula to be developed in what follows. Let us abbreviate

$$H(z) = H(0, [0, z])$$

$$= \sum_{n \geqq 0} Pr\{N > n \text{ and } Y_1 + \ldots + Y_n \leqq z\}.$$

Throughout, we assume that r(z) *is a monotonically nonincreasing function of* z. In words, the probability of failure $1 - r(z)$ increases with damage, a reasonable assumption. To avoid trivialities we assume $r(z) < 1$ for some z. Under this assumption, $w(x)$ is not only finite, but bounded as a function of x. To see this, take L such that $\beta = r(L) < 1$. Then, since the damages Y_k are nonnegative, an induction establishes $K^n(x, [0, \infty)) = K^n(x, [L, \infty)) \leqq \beta^n$ for $x \geqq L$, so that $w(x) \leqq [\lambda(1-\beta)]^{-1}$ in this region. On the other hand, for $x < L$, $w(x)$ can increase by no more than λ^{-1} times the mean number of Y_k's needed to achieve $Y_1 + \ldots + Y_n \geqq L$. That is, $w(x) \leqq \lambda^{-1}[M(L) + (1-\beta)^{-1}]$ for all x.

We will show that optimal Markov times take the form

$$T_\xi = \inf\{t \geqq 0;\ X(t) \geqq \xi\},$$

where ξ is fixed and nonnegative. (Here, as elsewhere in similar definitions, assign $T_\xi = \zeta$ whenever the set in braces is empty.) We will need two properties of such Markov times, both of which are expressible through the kernels K and H. Since each $n < N$ for which $x + Y_1 + \ldots + Y_n < \xi$ corresponds to an intershock time included in T_ξ, we have

$$E_x[T_\xi] = \lambda^{-1} H(x, [0, \xi)), \tag{2.7}$$

and

$$E[T_\xi] = \lambda^{-1} H(\xi-).$$

Next, by partitioning the event $\{T_\xi < \zeta\}$ according to the n for which $x+Y_1+\ldots+Y_n < \xi \leqq x+Y_1+\ldots+Y_n+Y_{n+1}$ we obtain

$$\begin{aligned} Pr\{T_\xi < \zeta\} &= \sum_{n=0}^{\infty} \int_0^\xi K^n(0, dz) \left\{ \int_{\xi-z}^{\infty} r(z+y) dF(y) \right\} \\ &= \int_0^\xi \left\{ \int_{\xi-z}^{\infty} r\ (z+y) dF(y) \right\} dH(z). \end{aligned} \tag{2.8}$$

Here is a final close similarity with renewal theory. By segregating the first term and factoring one application of K in the remainder of

$$w(x) = \lambda^{-1} \sum_{n=0}^{\infty} K^n(x, [0, \infty)),$$

we obtain

$$\begin{aligned} w(x) &= \lambda^{-1} + \lambda^{-1} \sum_{n=1}^{\infty} \int K(x, dz) K^{n-1}(z, [0, \infty)) \\ &= \lambda^{-1} + \int w(x+y) r(x+y) dF(y). \end{aligned} \tag{2.9}$$

Alternatively, (2.9) can be established by conditioning on the first intershock time and applying a standard renewal argument.

Since $w(x)$ is bounded we may apply the infinitesimal operator G. A comparison of (2.1) and (2.9) shows

$$Gw(x) = -1. \tag{2.10}$$

In fact, this result is a general one in the theory of Markov processes, as using (2.10), with $T = \zeta$, in Dynkin's formula will show.

3. OPTIMAL PLANNED REPLACEMENT

We continue with $\{X(t);\ 0 \leqq t < \zeta\}$ being the Markov process described in the previous section, interpreted as the cumulative damage process up to the failure time ζ of some machine or production system. We allow a controller to institute a *planned replacement* at any Markov time $T < \zeta$. He must replace upon system failure. Every replacement costs c dollars, and a *failure replacement*, the event $\{T = \zeta\}$, invokes an additional penalty of K dollars. If each replacement is with a new system having statistical properties identical with the original, and the replacement cycles are repeated indefinitely, then the law of large numbers tells us the long run average cost per unit time will be the expected cost

over a replacement cycle divided by the expected time between replacements. Note that both these expected values are finite. That is, the average cost associated with a control policy or Markov time T will be

$$\psi_T = \frac{c + K\,Pr\{T = \zeta\}}{E[T]}. \tag{3.1}$$

Let $\psi^* = \inf_T \psi_T > 0$ be the optimal average cost. In this section we show that an optimal Markov time T^* is determined by a single critical number ξ^*. The optimal strategy calls for replacement at failure, or when the cumulative damage $X(t)$ first exceeds ξ^*, whichever occurs first. That is

$$T^* = \inf\{t \geqq 0 : X(t) \geqq \xi^*\}. \tag{3.2}$$

Furthermore we will show that ξ^* and ψ^* may be determined by jointly solving

$$\xi^* = \inf\{x \geqq 0;\ \psi^* - \lambda K[1 - R(x)] \leqq 0\}, \tag{3.3}$$

where

$$R(x) = \int r(x+y)\,dF(y) \tag{3.4}$$

and

$$\begin{aligned} 0 &= \psi^* E[T^*] - c - K\,Pr\{T^* = \zeta\} \\ &= \psi^* \lambda^{-1} H(\xi^* -) - c - K\left\{1 - \int_0^{\xi^*} \int_{\xi^* - z}^{\infty} r(z+y)\,dF(y)\,dH(z)\right\}. \end{aligned} \tag{3.5}$$

To begin the derivation, note that for every Markov time T,

$$\psi^* \leqq \frac{c + K\,Pr\{T = \zeta\}}{E[T]} \tag{3.6}$$

and so by using $\psi^* > 0$,

$$\psi^* E[T] - c - K\,Pr\{T = \zeta\} \leqq 0. \tag{3.7}$$

In addition, equality holds in (3.6) if and only if equality holds in (3.7). That is, a Markov time T minimizes the average cost if and only if T maximizes

$$\theta_T = \psi^* E[T] - c - K\,Pr\{T = \zeta\} \tag{3.8}$$

and the maximum value of θ_T is zero. We have

$$E[T] = E[\zeta] - E[\zeta - T;\ T < \zeta]. \tag{3.9}$$

Since the controlled process begins with no damage $E[\zeta]=w(0)$ where $w(x)=E[\zeta|X(0)=x]$ is given by Equation (2.6), and conditioning on $X(T)$ with the strong Markov property shows

$$E[\zeta-T;\, T<\zeta]=E[w(X(T));\, T<\zeta]. \tag{3.10}$$

Also write

$$K\,Pr\{T=\zeta\}=K-E[K;\, T<\zeta]. \tag{3.11}$$

Then

$$\theta_T=-c-\{E[\phi(X(T));\, T<\zeta]-\phi(0)\}, \tag{3.12}$$

where

$$\phi(x)=\psi^* w(x)-K. \tag{3.13}$$

Since $w(x)$ is bounded as a function of x we may use Dynkin's formula to express

$$\theta_T=-c+E\left[\int_0^T g(X(s))\,ds\right], \tag{3.14}$$

where

$$\begin{aligned} g(x)&=-G\phi(x)\\ &=-G(\psi^* w(\cdot)-K)(x)\\ &=\psi^*-\lambda K(1-R(x)). \end{aligned} \tag{3.15}$$

We referred to (2.1), (2.10), and (3.4) in computing $g(x)$.

As now expressed, our problem is that of maximizing (3.14), where ψ^* is chosen so that the maximum has value 0.

By its definition in (3.3), ξ^* is the smallest value x for which $g(x)$ is nonpositive. That is

$$g(x)>0 \qquad \text{for } x<\xi^* \tag{3.16}$$

and

$$g(x)\leqq 0 \qquad \text{for } x\geqq\xi^*.$$

Since T^* is defined to be the smallest t for which $X(t)\geqq\xi^*$, we have

$$g(X(t))>0 \qquad \text{for } t<T^*. \tag{3.17}$$

Because of the assumption that the damages $Y_1, Y_2, \ldots$ are nonnegative, once the $X(t)$ process exceeds ξ^* it remains in this region thereafter. Thus we obtain the converse to (3.17),

(3.18) $$g(X(t)) \leqq 0 \qquad \text{for } t \geqq T^*.$$

This should make it clear that T^* maximizes (3.14). We need only write, for any Markov time T

$$\theta_T - \theta_{T^*} = E\left[\int_0^{T^*} g(X(s))ds\right] - E\left[\int_0^{T} g(X(s))ds\right]$$

$$= E\left[\int_T^{T^*} g(X(s))ds;\, T < T^*\right] - E\left[\int_{T^*}^{T} g(X(s))ds;\; T^* \leqq T\right]$$

$$\geqq 0.$$

It remains to impose the boundary condition $\theta_{T^*}=0$. By use of (2.7) and (2.8), this is expressed in (3.5), and the proof of optimality is complete.

The stopping problem considered here has a structure similar to, but outside the scope of, stopping problems treated by Ross [14].

4. A SPECIAL CASE—THE THRESHOLD MODEL

The cumulative damage failure model that has received the most attention in the literature employs the survivorship function

(4.1) $$r(z) = \begin{cases} 1 & \text{for } 0 \leqq z < L, \\ 0 & \text{for } L \leqq z. \end{cases}$$

In words, system failure occurs when the cumulative damage first exceeds a fixed threshold L. Section 1.10 of Buckland [4] contains a review of work on this failure model.

In this threshold situation, several of the important quantities appearing in Equations (3.3) and (3.5) that describe the optimal replacement strategy can be phrased in terms of the renewal function

(4.2) $$M(z) = \sum_{n=0}^{\infty} Pr\{Y_1 + \cdots + Y_n \leqq z\}$$

$$= \sum_{n=0}^{\infty} F^{(n)}(z)$$

associated with the damage distribution $F(z)=Pr\{Y_k \leqq z\}$. Here we use $F^{(n)}$ to denote the n-fold convolution of F. Indeed,

(4.3) $$H(z) = H(0, [0, z])$$
$$= \begin{cases} M(z) & \text{for } z \leqq L \\ M(L) & \text{for } z > L. \end{cases}$$

Since

(4.4) $$R(x) = \int r(x+y)dF(y) = F((L-x)^+),$$

according to (3.3), ξ^* is given by

$$\xi^* = \inf\{x \geqq 0;\ \psi^* - \lambda K[1 - F((L-x)^+)] \leqq 0\}. \tag{4.5}$$

Writing $z \wedge L = \min\{z, L\}$, Equation (3.5), the second of our equation pair determining ξ^* and L^*, becomes

$$0 = \psi^* \lambda^{-1} M(L \wedge \xi^* -) - c - K\left\{1 - \int_0^{L \wedge \xi^*} [F(L-z) - F(\xi^* - z)]\, dM(z)\right\}. \tag{4.6}$$

Especially simple formula result when F is the exponential distribution

$$F(y) = 1 - e^{-\mu y} \qquad \text{for } y \geqq 0.$$

The renewal function in this case, as everyone knows, (Problem 8, p. 215 Feller [12]) is simply

$$M(z) = 1 + \mu z. \tag{4.7}$$

Furthermore, the renewal equation,

$$M(z) = 1 + \int_0^z F(z-y)\, dM(y),$$

can be rearranged to give

$$\int_0^z \{1 - F(z-y)\}\, dM(y) = 1, \tag{4.8}$$

whence

$$\begin{aligned} \int_0^{\xi^*} [F(L-z) - F(\xi^* - z)]\, dM(z) &= \int_0^{\xi^*} \{1 - F(\xi^* - z)\}\, dM(z) - \int_0^{\xi^*} \{1 - F(L-z)\}\, dM(z) \\ &= 1 - e^{-\mu(L-\xi^*)} \int_0^{\xi^*} \{1 - F(\xi^* - z)\}\, dM(z) \\ &= 1 - e^{-\mu(L-\xi^*)}. \end{aligned} \tag{4.9}$$

Thus, our two Equations (4.5) and (4.6) reduce in this case to simply

$$\psi^* = \lambda K e^{-\mu(L-\xi^*)} \tag{4.10}$$

and

$$0 = \psi^* \lambda^{-1}(1 + \mu\xi^*) - c - K e^{-\mu(L-\xi^*)}. \tag{4.11}$$

Using (4.10), the second equation simplifies to

$$\psi^* \xi^* = c\lambda/\mu. \tag{4.11a}$$

Note that the units for cost, time, and damage are free to be chosen. With this in mind, we introduce the dimensionless variables

$$\xi_0^* = \mu\xi^*, \qquad L_0 = \mu L$$
$$\psi_0^* = \psi^*/(K\lambda).$$

Then (4.10) and (4.11a) become

$$\psi_0^* = \exp\{-(L_0 - \xi_0^*)\}$$

and

$$c/K = \xi_0^* \Psi_0^*.$$

By eliminating Ψ_0^*, we obtain

$$c/K = \xi_0^* e^{-(L_0 - \xi_0^*)}.$$

Thus the optimal strategy simply depends on the ratio of costs c/K and the normalized threshold L_0.

5. MORE GENERAL COST AND INCOME MODELS

In section 3 it was implicitly assumed that replacements were instantaneous. If the act of repairing or replacing the system requires a significant amount of time, the model should include the factor of income lost during this period. Let I be the mean rate of income per unit time when the system is operating, let τ_f be the down time associated with a failure replacement and let τ_p be the down time associated with a planned replacement. We now attempt to maximize the long run net income per unit time, which, as before, is the expected net income per replacement cycle divided by the expected duration between replacements. Continuing our previous notation for the previously defined quantities, this net income is

$$\Delta_T = \frac{IE[T] - c - K\,Pr\{T = \zeta\}}{E[T] + \tau_p Pr\{T < \zeta\} + \tau_f Pr\{T = \zeta\}}. \tag{5.1}$$

Let $\Delta^* = \sup_T \Delta_T$ be the maximal income rate. We may assume $\Delta^* > 0$, since otherwise it would not be profitable to operate at all. We argue as we did earlier to conclude that T maximizes Δ_T if and only if T maximizes θ_T, and the maximum value of the latter is $\sup \theta_T = 0$, where now

$$\theta_T = IE[T] - c - K\,Pr\{T = \zeta\} - \Delta^*\{E[T] + \tau_p Pr\{T < \zeta\} + \tau_f Pr\{T = \zeta\}\}.$$

Reduced to,

$$\theta_T = (I - \Delta^*)E[T] - (c + \Delta^*\tau_p) - (K + \Delta^*\tau_f - \Delta^*\tau_p)Pr\{T = \zeta\}, \tag{5.2}$$

this criterion has the same form as θ_T given in (3.8). We conclude the optimal strategy calls for replacement at failure, or when the cumulative damage $X(t)$ first exceeds a critical number ξ^*, whichever occurs first. We compare (3.8) and (5.2) to learn that ξ^* and Δ^* are determined by Equations (3.3) and (3.5) modified by replacing

$$\psi^* \quad \text{with } I-\Delta^*,$$

$$c \quad \text{with } c+\Delta^*\tau_p,$$

and

$$K \quad \text{with } K+\Delta^*\tau_f-\Delta^*\tau_p.$$

That is, ξ^* and Δ^* are obtained by jointly solving

$$\xi^*=\inf\{x \geqq 0;\ I-\Delta^*-\lambda(K+\Delta^*\tau_f-\Delta^*\tau_p)[1-R(x)] \leqq 0\}$$

and

$$0=(I-\Delta^*)\lambda^{-1}H(\xi^*-)-(c+\Delta^*\tau_p)-(K+\Delta^*\tau_f-\Delta^*\tau_p)\left\{1-\int_0^{\xi^*}\int^{\infty} r(z+y)dF(y)dH(z)\right\}.$$

6. A MORE GENERAL FAILURE MODEL

We continue to assume that shocks occur to a machine or production system in accordance with a Poisson process, denoted $\{N(t);\ 0 \leqq t < \infty\}$, having a known rate λ. However, now let us assume that the system survives at least k shocks with an arbitrary but known probability P_k, for $k=0, 1, \ldots$. The additive damage model described in section 2 fits this framework with $p_k = Pr\{N > k\} = K^k(0, [0, \infty))$, where $N=N(\zeta)$ is the number of shocks at failure. Now, however, we are allowing $\{P_k\}$ to be any decreasing sequence of numbers in the unit interval, subject only to

$$\sum_{k=0}^{\infty} P_k < \infty. \tag{6.1}$$

We suppose that the shocks, when they occur, are observable by the controller, and thus his replacement decisions may be based on the number of shocks that have occurred. We will show that without loss in generality we need only consider policies that replace either upon failure or when the cumulative number of shocks first reaches a control limit A, and we will express the long run average cost per unit time as a simple function of A so that the optimal control limit A^* may be computed easily.

Let ζ be the time that the system fails. By conditioning on the number of shocks in $(0, t]$ it is easy to compute the system survival probability

$$Pr\{\zeta > t\} = \sum_{k=0}^{\infty} \frac{(\lambda t)^k e^{-\lambda t}}{k!} P_k, \tag{6.2}$$

and the mean time to failure

$$E[\zeta] = \int_0^{\infty} Pr\{\zeta > t\}dt$$

$$= \lambda^{-1} \sum_{k \geqq 0} P_k.$$

It follows from the assumption expressed in (6.1) that $E[\zeta] < \infty$. We remark in passing that the process $\{N(t);\ 0 \leqq t < \zeta\}$ is a Markov process, and, to be precise, is a terminating pure birth process.

We allow the controller to institute a planned replacement at any Markov time $T < \zeta$. Every replacement costs c dollars, and a failure in service, the event $\{T \geqq \zeta\}$ invokes an additional penalty of K dollars. As before, each replacement is with a new system having properties that are statistically equivalent to the original, and the replacement cycles are repeated indefinitely. Thus the long run average cost per unit time will be the expected cost over a replacement cycle divided by the expected time between replacements. Note that these expected values are finite by (6.2). That is, the average cost associated with a control policy or Markov time T will be

$$\psi_T = \frac{c + K\,Pr\{T = \zeta\}}{E[T]}. \tag{6.3}$$

The theory of optimal stopping in a Markov process (Theorem 3 of Taylor [15]) implies that when a Markov time T^* minimizing ψ_T exists, it may be taken as the entry time to a set of states. Since our pure birth process $\{N(t);\ 0 \leqq t < \zeta\}$ moves through the states one-by-one in increasing order, the process first reaches a given set of states when it first reaches the smallest state in the set. That is, without loss of generality, we may restrict our attention to Markov times of the form

$$T(A) = \inf\{t \geqq 0 : N(t) \geqq A\}.$$

Now let us observe that the $N(t)$ process reaches A before failing if and only if it survives the first A shocks. That is,

$$Pr\{\zeta \leqq T(A)\} = 1 - Pr\{T(A) < \zeta\}$$

$$= 1 - P_A.$$

Again, the process spends a mean time of λ^{-1} in every state it reaches. Weighing this by the probability of reaching the various states k and adding for $k = 0, 1, \ldots, A-1$ we obtain

$$E[T(A)] = \lambda^{-1} \sum_{k=0}^{A-1} P_k.$$

Thus, under the control rule $T(A)$, the average cost expressed in (6.3) is

$$\psi_{T(A)} = \frac{c + K(1 - P_A)}{\lambda^{-1} \sum_{k=0}^{A-1} P_k}. \tag{6.4}$$

One can then numerically search for the $A = A^*$ that minimizes $\psi_{T(A)}$.

Note: This shock model is a special case of the additive damage process $\{X(t);\ 0 \leqq t < \zeta\}$ of section 2. To see this set $N(t) = X(t)$ where $Y_1 = Y_2 = \ldots = 1$, and $r(z) = P_{[z]}/P_{[z-1]}$, $z \geqq 1$. What is new in the current section is that we do *not* require the conditional survivorship $P_{[z]}/P_{[z-1]}$ to be monotonic. If it happens to be, we may apply the analysis of section 3.

7. A RELATED STOPPING PROBLEM.

Let α and β be fixed real numbers, with $\beta > 0$. Our task in this section is to find a Markov time T that maximizes

$$E[\alpha + \beta X(T); T < \zeta], \tag{7.1}$$

where $\{X(t); 0 \leqq t < \zeta\}$ is the threshold process in which $r(z)$ is 1 or 0 according as $z < L$ or $z \geqq L$. Equivalently, we desire a Markov time T that maximizes $E[u(X(T)); T < \zeta]$ where

$$u(x) = \begin{cases} \alpha + \beta x & \text{for } 0 \leqq x < L, \\ 0 & \text{for } L \leqq x. \end{cases}$$

PROPOSITION 1: *There exists a value* $A^* \in [0, L]$ *for which the Markov time,*

$$T^* = \inf\{t \geqq 0 : X(t) \geqq A^*\}$$

maximizes (7.1).

(a) *If* $\alpha \leqq -\beta L$, *take* $A^* = L$. *The corresponding optimal value of* (7.1) *is zero.*

(b) *If* $\alpha[1 - F(L)] - \beta \int_0^L t dF(t) \geqq 0$, *take* $A^* = 0$. *The corresponding optimal value of* (7.1) *is* $\alpha \geqq 0$.

(c) *Otherwise, take* $A^* \in (0, L)$ *as the unique solution to* $g(A^*) = 0$ *where*

$$g(A) = \lambda\{(\alpha + \beta A)[1 - F(L - A)] - \beta \int_0^{L-A} t dF(t)\}. \tag{7.2}$$

The corresponding optimal value of (7.1) *at* $X(0) = x$ *is*

$$f(x) = \begin{cases} \int_{[0, A^* - x)} \int_{A^* - x - y}^{L - x - y} [\alpha + \beta(x + y + z)] dF(z) M[dy] & \text{for } 0 \leqq x < A^*, \\ \alpha + \beta x \qquad \text{for } A^* \leqq x < L, \end{cases} \tag{7.3}$$

where M *is the renewal measure corresponding to* F.

PROOF: Since u is bounded, it is in the domain of the infinitesimal operator G. We compute

$$\begin{aligned} g(x) &= -Gu(x) \\ &= \lambda\left\{(\alpha + \beta x)[1 - F(L - x)] - \beta \int_0^{L-x} y dF(y)\right\} \end{aligned} \tag{7.4}$$

by using (2.1) and the threshold form of r. From Dynkin's formula,

$$E[u(X(T)); T < \zeta] - u(0) = E\left[\int_0^T Gu(X(s)) ds\right].$$

Thus, using the same technique as before, we need only show that $g(x) = -Gu(x)$ satisfies

$$g(x) < 0 \qquad \text{for } x < A^* \tag{7.5}$$

and

$$g(x) \geqq 0 \qquad \text{for } x \geqq A^*.$$

We will show first that the three cases $(a) - (c)$ are mutually exclusive and totally exhaustive. After this, we will prove the optimality only in case (c), leaving the other two easier cases to the reader.

Case (a) corresponds to $g(L) \leqq 0$, where $g(A)$, $0 \leqq A \leqq L$ is given in (7.4). Let us consider, then, the case $g(L) = \alpha + \beta L > 0$. Observe from the defining equation (7.4) that $g(A)$ is strictly increasing in A wherever $g(A) \geqq 0$. It follows that there is a unique value of A, namely

$$A^* = \inf \{x \geqq 0 : g(x) \geqq 0\}$$

for which (7.4) holds. Case (b) corresponds to $g(0) \geqq 0$, and then $A^* = 0$. Case (c) corresponds to $g(0) < 0$. Since we have supposed $g(L) > 0$, $0 < A^* < L$ in this case. The solution is unique by the strictly increasing property of g just mentioned. This concludes the proof.

8. MAXIMIZING THE DURATION OF OPERATION

Let $\{X(t); 0 \leqq t < \zeta\}$ be the threshold process in which $r(x)$ is 1 or 0 according as $x < L$ or $x \geqq L$. Let a and b be fixed numbers with $b > 0$. In this section we find a Markov time T^* that maximizes

$$\Lambda_T = E[a + bT; T < \zeta]. \tag{8.1}$$

In this problem we receive a fixed reward a together with a reward per unit time at rate b, *provided that we replace before failure*, and nothing otherwise.

The optimal Markov time here is more complicated than the solution to our earlier problems. Let

$$t_0 = \frac{1}{\lambda\{1 - F(L)\}} - \frac{a}{b} \tag{8.2}$$

and

$$t_L = \lambda^{-1} - a/b. \tag{8.3}$$

For $t_L \leqq t \leqq t_0$ and $0 \leqq x < L$ define

$$g(t, x) = b - \lambda(a + bt)[1 - F(L - x)]. \tag{8.4}$$

We will assume $t_0 > 0$, since otherwise it is optimal to stop immediately and take a reward of a, and then define

$$c(t) = \begin{cases} L & \text{for } t \leqq t_L, \\ \inf \{x: g(t, x) = 0\} & \text{for } t_L < t < t_0, \\ 0 & \text{for } t \geqq t_0. \end{cases} \tag{8.5}$$

We will show that the optimal Markov time is

$$T^* = \inf\{t \geqq 0: X(t) \geqq c(t)\}. \tag{8.6}$$

Let us begin by noting that the bivariate process $\{(t, X(t)); 0 \leqq t < \zeta\}$ is a Markov process whose infinitesimal operator G coincides on functions $f(t, x)$ that are bounded and sufficiently smooth with the operator

$$Gf(t, x) = f_t(t, x) - \lambda\{f(t, x) - \int_0^{L-x} f(t, x+y)dF(y)\}, \tag{8.7}$$

where $f_t = \partial f/\partial t$. Let us suppose for the moment that the function

$$\tau(t, x) = a + bt \qquad \text{for } t \geqq 0,\ 0 \leqq x < L \tag{8.8}$$

is in the domain of G; it isn't, of course, because it is not bounded, but let us proceed as if it were. Let T be any Markov time and note that the usual convention $T \leqq \zeta$ implies that all moments of T are finite. Then Dynkin's formula (Theorem 5.1 and its corollary, Dynkin [11]) would apply to give

$$E_x[\tau(T, X(T)); T < \zeta] - \tau(0, x) = E_x\left[\int_0^T G\tau(s, X(s))ds\right]. \tag{8.9}$$

By using (8.7) and (8.8), we compute

$$\begin{aligned} G\tau(t, x) &= b - \lambda\{(a+bt)[1-F(L-x)]\} \\ &= g(t, x), \end{aligned}$$

where g is defined in (8.4), so that (8.9) becomes

$$\begin{aligned} \Lambda_T &= E[a + bT; T < \zeta] \\ &= a + E\left[\int_0^T g(s, X(s))ds\right]. \end{aligned} \tag{8.10}$$

The crucial point is that by the definitions of g, c, and T^* given in (8.4)–(8.6), we have

$$g(s, X(s)) > 0 \qquad \text{for } s < T^*$$

and

$$g(s, X(s)) < 0 \qquad \text{for } s > T^*.$$

It is important in drawing this conclusion to note that once the $X(t)$ process has crossed the boundary $c(t)$, it can never go back. By using (8.10) we have

$$\begin{aligned} \Lambda_{T^*} - \Lambda_T &= E\left[\int_0^{T^*} g(s, X(s))ds\right] - E\left[\int_0^T g(s, X(s))ds\right] \\ &= E\left[\int_T^{T^*} g(s, X(s))ds;\ T < T^*\right] - E\left[\int_{T^*}^T g(s, X(s))ds;\ T^* \leqq T\right] \\ &\geqq 0 \end{aligned}$$

and thus

$$\Lambda_{T^*} \geqq \Lambda_T,$$

which proves that T^* is optimal. But the argument depends on (8.10)

$$E[a+bT;\ T<\zeta] = a+E\left[\int_0^T g(s,X(s))ds\right],$$

whose derivation, as given, was faulty. Here is the correction. For any $q>0$ define

$$\tau_q(t,x) = a+bq(1-e^{-t/q}).$$

Observe that τ_q is bounded and smooth, hence in the domain of G, that

$$\lim_{q\to\infty}\tau_q(t,x) = a+bt$$

and that the limit is monotone. Then

$$\begin{aligned} g_q(t,x) &= G\tau_q(t,x) \\ &= be^{-t/q} - \lambda[a+bq(1-e^{-t/q})][1-F(L-x)], \end{aligned}$$

and Dynkin's formula gives

$$E[\tau_q(T,X(T));\ T<\zeta] = a+E\left[\int_0^T g_q(s,X(s))ds\right].$$

Now, as $q\to\infty$, the left side of (8.11) converges to $\Lambda_T = E[\tau(T,X(T));\ T<\zeta]$ by monotone convergence, while the right side converges to $a+E\left[\int_0^T g(s,X(s))ds\right]$ by dominated convergence, a dominating integrable random variable being

$$\int_0^\zeta [b+\lambda(|a|+bs)]ds.$$

The result is (8.10). The proof of the optimality of T^* is complete.

9. THE ECONOMICS OF INFORMATION

The following rules require differing amounts of information, and it is interesting to ask what this information buys in terms of lower operating costs:

1. Repair when the damage first exceeds ξ;
2. Repair when A shocks have occurred;
3. Repair at time B;

To illustrate the point, C. Macken has computed for us the optimal control limits and minimum average

costs of a specific system controlled by these rules. Both the intershock and the damage distributions were exponential with unit mean, and the threshold failure model with threshold $L=5$, was used. The following table gives the results, with $c=1$ and both $K=1$ and 10.

Rule		$K=1$	$K=10$
	ξ^*	3.69	2.00
1	Average cost	0.27	0.50
	Probability of failure	0.27	0.050
	Mean time to repair	4.69	3.00
	A^*	6	2
2	Average cost	0.32	0.70
	Probability of failure	0.62	0.040
	Mean time to repair	5.12	1.99
	B^*	7.99	2.11
3	Average cost	0.33	0.96
	Probability of failure	0.76	0.096
	Mean time to repair	5.36	2.04

When the failure cost K equals the repair cost c, the average costs per unit time range from 0.27 up to 0.33, and the simple rule of repairing at fixed intervals exceeds the minimum cost by 22 percent. When failure in service is more traumatic, and $K=10$, the average costs range from 0.50 to 0.96. The simple rule now is almost twice as costly as the rule based on the observed damage.

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LOWER CONFIDENCE LIMITS FOR THE IMPACT PROBABILITY WITHIN A CIRCLE IN THE NORMAL CASE*

S. Zacks

Case Western Reserve University

H. Solomon

Stanford University

ABSTRACT

Lower confidence limits are derived for the impact probability within a circle of fixed radius. The bivariate normal distribution with zero means, unequal variances, and zero correlation is the probability model for impacts. A new representation of the impact probability function is offered. This presentation is valid also for the dependent case, where the eigenvalues of the covariance matrix replace the variances. When the ratio of variances is known the lower confidence limits are uniformly most accurate (UMA). A few alternative approaches are compared by simulation when the ratio of variances is unknown.

1. INTRODUCTION

Let (X, Y) denote the coordinates of an impact point. We assume that X and Y are independent random variables each normally distributed with zero means and variances σ^2 and v^2, respectively. It is further assumed that $v^2 \geq \sigma^2$. That is, we know a priori that the variance of Y is not smaller than that of X. We are interested in statements about the probability that (X, Y) lies within a circle, centered at the origin with a specified radius, R. This probability, called the impact probability, is given by the function

$$P(R; \sigma, v) = P_{\sigma, v}[X^2 + Y^2 \leq R^2]. \tag{1.1}$$

In section 2 we show that this impact probability function can be presented as a mixture of tail probabilities of Poisson distributions, where the mixing probabilities are given in terms of a negative-binomial distribution. This presentation can be derived from earlier results of Pitman and Robbins [8], Ruben [9], Neuts and Zacks [7] and others. A recent paper of Gilliland and Hansen [4] reviews prior results and provides an appropriate mixture representation for (1.1) when the mean vector is not zero. Some authors have represented the impact probability as an integral of the modified Bessel function $I_0(x)$ (see e.g., DiDonato and Jarnagin [2]). For statistical purposes it is more expeditious to treat this probability as a mixture of Poisson probabilities.

The main objective of the present paper is to develop an efficient lower confidence limit for the impact probability. This problem has not been discussed directly in the literature. Inselmann and

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Granville [5] have derived the uniformly minimum variance unbiased (UMVU) estimator of the impact probability when $\sigma^2 = v^2$. The UMVU estimator of the impact probability under such a restricted model ($\sigma^2 = v^2$) is a special case of the general problem of UMVU estimators of the c.d.f. of gamma distributions (see Zacks [10, p. 111]). In the general case, when $v^2 \geq \sigma^2$, the UMVU is somewhat more complicated, but can be easily derived. Blischke [1] has derived the asymptotic variance of the maximum likelihood estimator of the CEP (circular probable error) for the bivariate normal distribution not centered at the origin. In section 5 similar methods are employed to handle our problem.

The problem of deriving lower confidence limits when the ratio of variances $\tau^2 = v^2/\sigma^2$ is known is not difficult. In section 3 we derive the uniformly most accurate (UMA) lower confidence limits for cases of known τ^2. When both σ^2 and τ^2 are unknown the problem becomes a difficult one. One can attempt to use the Bonferroni inequality but, as will be shown in section 6, a proper lower confidence limit obtained that way is too conservative; its actual coverage probability is far above the nominal one.

In order to obtain lower confidence limits whose coverage probabilities are close to the nominal confidence probability, the relationship between a simultaneous $(1-\alpha)$ confidence region in the (τ^2, σ^2)-plane and contours of equal impact probability corresponding to points on the boundary of that region is analyzed in section 4. This analysis leads to the consideration of three lower limits, which are called approximate confidence limits. In section 5 we develop a lower limit based on the asymptotic normal distribution of the maximum likelihood estimator (MLE) of the impact probability. This lower limit is an asymptotically UMA confidence limit. One of the approximate confidence limits proposed in section 4, which is the limit derived in section 3 in which an MLE of τ^2 is substituted is also an asymptotically UMA confidence limit.

One question is how the one based on the MLE fares in small samples. In order to compare the characteristics of the approximate confidence limits in small sample situations we present the results of simulation runs, based on samples of size $n = 30$ in section 6. These simulations indicate clearly that the two asymptotically UMA confidence limits provide similar results in samples of size 30. Furthermore, the estimated coverage probabilities and expected shortages of the asymptotically UMA confidence limit of section 4, show that it provides a good estimation procedure over a wide combination of σ^2 and τ^2 values. This approximate confidence limit is therefore recommended since it is relatively simpler for calculations than the other competitive limits.

2. THE IMPACT PROBABILITY FUNCTION

We develop now a formula for the impact probability starting with the case of independence. It is known (Neuts and Zacks [7]) that if X and Y are independent random variables, $X^2 \sim \sigma^2\chi^2[1]$ and $Y^2 \sim \sigma^2\tau^2\chi^2[1]$ then

$$X^2 + Y^2 \sim \sigma^2\chi^2[2+2J], \tag{2.1}$$

where J is a random variable having a Negative Binomial distribution. The probability function of J is

$$g(j|\theta) = \frac{\Gamma\left(j+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(j+1)}(1-\theta)^{1/2}\theta^j, \quad j = 0, 1, \ldots, \tag{2.2}$$

$$\text{where } \theta = 1 - \frac{1}{\tau^2}, \qquad \tau^2 = v^2/\sigma^2,$$

and $\chi^2[\nu]$ denotes a central chi-square random variable with ν degrees of freedom. Accordingly, the impact probability is

$$P[X^2 + Y^2 \leq R^2] = \sum_{j=0}^{\infty} g(j|\theta) P\left\{\chi^2[2+2j] \leq \left(\frac{R}{\sigma}\right)^2\right\}. \tag{2.3}$$

Furthermore, for every $j = 0, 1, \ldots$

$$P\left\{\chi^2[2+2j] \leq \left(\frac{R}{\sigma}\right)^2\right\} = 1 - \text{Pos}\left(j \,\middle|\, \frac{1}{2}\left(\frac{R}{\sigma}\right)^2\right), \tag{2.4}$$

where Pos $(j|\lambda)$ is the Poisson cumulative distribution function (c.d.f.) with mean λ. Accordingly, we define the function

$$\psi(x, \theta) = 1 - E_\theta\left\{\text{Pos}\left(J \,\middle|\, \frac{1}{2}x\right)\right\}, \tag{2.5}$$

for $0 < x < \infty$ and $0 < \theta < 1$, where the expectation in (2.5) is with respect to the Negative-Binomial given by (2.2). Hence, the impact probability in the independent case is

$$P(R; \sigma, v) = \psi\left[\left(\frac{R}{\sigma}\right)^2, \left(1 - \frac{\sigma^2}{v^2}\right)\right], 0 < \sigma \leq v < \infty. \tag{2.6}$$

3. UMA LOWER CONFIDENCE LIMITS WHEN THE VARIANCE RATIO τ^2 IS KNOWN

For a sample of n impact points (X_i, Y_i), $i = 1, 2, \ldots n$; let $Q_1 = \sum_{i=1}^{n} X_i^2$ and $Q_2 = \sum_{i=1}^{n} Y_i^2$. When τ^2 is known, a sufficient statistic is $T = Q_1 + Q_2/\tau^2$ and T is distributed as $\sigma^2\chi^2[2n]$. A UMA upper confidence limit, at level $1-\alpha$, for σ^2 is given by (Zacks [7, p. 453])

$$\bar{\sigma}_\alpha^2 = T/\chi_\alpha^2[2n], \tag{3.1}$$

where $\chi_\alpha^2[2n]$ is the α-fractile of $\chi^2[2n]$.

The family of Poisson distributions has the monotone likelihood ratio (MLR) property. Therefore, if $\eta_1 < \eta_2$ then

$$\text{Pos}(J|\eta_1) \geq \text{Pos}(J|\eta_2) \ \ldots \text{ for every } J = 0, 1, \ldots . \tag{3.2}$$

Hence, if $\sigma^2 \leq \bar{\sigma}_\alpha^2$ then, for each $1 < \tau^2 < \infty$,

$$E_\theta\left\{\text{Pos}\left(J \,\middle|\, \frac{1}{2}\left(\frac{R}{\bar{\sigma}_\alpha}\right)^2\right)\right\} \geq E_\theta\left\{\text{Pos}\left(J \,\middle|\, \frac{1}{2}\left(\frac{R}{\sigma}\right)^2\right)\right\}, \tag{3.3}$$

where $\theta = 1 - \frac{1}{\tau^2}$, and the expectation is with respect to the Negative-Binomial distribution with p.d.f. (2.2).

It follows that

$$P_{\sigma^2}\left\{ \psi\left(\frac{R^2}{\bar{\sigma}^2_\alpha}, \theta \right) \leq \psi\left(\frac{R^2}{\sigma^2}, \theta \right) \right\} \geq \tag{3.4}$$

$$P_{\sigma^2}\{\sigma^2 \leq \bar{\sigma}^2_\alpha\} = 1 - \alpha, \text{ for } all\ 0 < \sigma < \infty.$$

This proves that $\psi\left(\left(\frac{R}{\bar{\sigma}_\alpha} \right)^2, \theta \right)$ is a lower confidence limit for the impact probability $P(R; \sigma^2, \tau)$. More explicitly, the lower confidence limit is

$$\psi\left(\frac{R^2}{\bar{\sigma}^2_\alpha}, \theta \right) = 1 - E_\theta\left\{ \text{Pos}\left(J \,\middle|\, \frac{1}{2} \cdot \frac{R^2 \chi^2_\alpha[2n]}{T} \right) \right\}. \tag{3.5}$$

The function $\psi\left(\frac{R^2}{\sigma^2}, \theta \right)$ is continuous in σ^2 for each θ. Furthermore, $T/\chi^2_\alpha[2n] \to \sigma^2$ a.s., as $n \to \infty$. This implies that the lower confidence limit (3.5) is a *strongly consistent* estimator of the impact probability.

We now prove that (3.5) is a UMA lower confidence limit. Let us start by showing that for each (fixed) θ, $\psi\left[\left(\frac{R}{\sigma} \right)^2, \theta \right]$ is a strictly decreasing function of σ^2. Indeed, for each $j = 1, 2, \ldots$

$$\frac{d}{d\lambda} \text{Pos}\,(j|\lambda) = -[\text{Pos}\,(j|\lambda) - \text{Pos}\,(j-1|\lambda)] = -p(j|\lambda), \tag{3.6}$$

where $p(j|\lambda)$ is the Poisson probability function with mean λ. Furthermore, differentiating $E_\theta\{\text{Pos}\,(J|\lambda)\}$ term by term we obtain

$$\frac{d}{d\lambda} E_\theta\{\text{Pos}\,(J|\lambda)\} < 0. \tag{3.7}$$

Finally, since $\frac{1}{2}\left(\frac{R^2}{\sigma^2}\right)$ is a strictly decreasing function of σ^2 we obtain from (2.5) that $\psi\left[\left(\frac{R}{\sigma}\right)^2, \theta\right]$ is a strictly decreasing function of σ^2. Thus, the correspondence between $P(R; \sigma, v)$ and σ^2, for fixed values of (R, v) is one-to-one. We proceed now to prove that $\psi\left(\frac{R^2}{\bar{\sigma}^2_\alpha}, \theta\right)$ is UMA.

If $\psi\left(\frac{R^2}{\bar{\sigma}^2_\alpha}, \theta\right)$ is not a UMA lower confidence limit, there exists another lower confidence limit, $\hat{\psi}$ say, such that for some $\psi_1 < \psi$

$$P_\psi\left[\psi\left(\frac{R^2}{\bar{\sigma}^2_\alpha}, \theta\right) \leq \psi_1\right] > P_\psi\left[\hat{\psi} \leq \psi_1\right]. \tag{3.8}$$

Let $\sigma^2(\psi)$ be the value of σ^2 corresponding to ψ for fixed θ. Due to the one-to-one correspondence between σ^2 and $\psi\left(\frac{R^2}{\sigma^2}, \theta\right)$, for each fixed θ, we obtain from (3.8) the inequality

$$P_{\sigma^2(\psi)}[\bar{\sigma}_\alpha^2 \geq \sigma^2(\psi_1)] > P_{\sigma^2(\psi)}[\sigma^2(\hat{\psi}) \geq \sigma^2(\psi_1)], \tag{3.9}$$

where $\sigma^2(\psi) < \sigma^2(\psi_1)$. Moreover, $\sigma^2(\hat{\psi})$ is an upper confidence limit for σ^2, at level $(1-\alpha)$, which by assumption is different from $\bar{\sigma}_\alpha^2$. But (3.9) contradicts the fact that $\bar{\sigma}_\alpha^2$ is a UMA upper confidence limit. This completes the proof.

4. LOWER CONFIDENCE LIMITS WHEN τ^2 IS UNKNOWN

In the present case the minimal sufficient statistic is (Q_1, Q_2). Let us construct a confidence region C_α in the (σ^2, v^2)-plane, of level $1-\alpha$. Since $\frac{Q_1}{\sigma^2} + \frac{Q_2}{v^2} \sim \chi^2[2n]$, we obtain

$$P_{\sigma^2, v^2}\left\{\frac{\hat{\sigma}_\alpha^2}{\sigma^2} + \frac{\hat{v}_\alpha^2}{v^2} \geq 1\right\} \geq 1-\alpha \tag{4.1}$$

for all $o < \sigma^2 \leq v^2 < \infty$, where $\hat{\sigma}_\alpha^2 = Q_1/\chi_\alpha^2[2n]$ and $\hat{v}_\alpha^2 = \max\left(\hat{\sigma}_\alpha^2, \frac{Q_2}{\chi_\alpha^2[2n]}\right)$. The confidence region C_α can be represented in the (τ^2, σ^2) plane as in Figure 1.

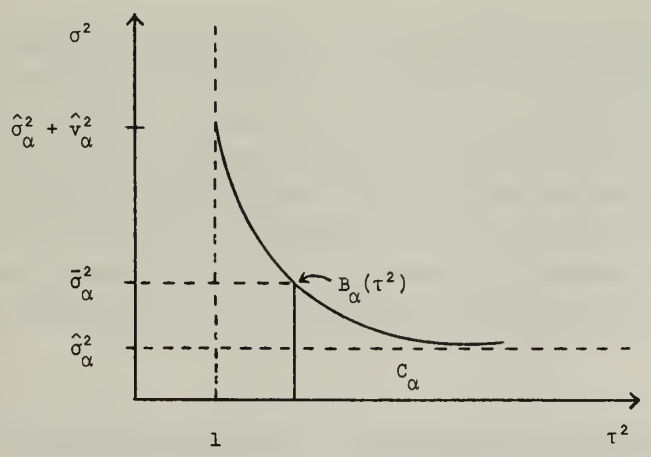

FIGURE 1. Simultaneous confidence region for (τ^2, σ^2)

In the previous section we have proven that the impact probability function is strictly decreasing with σ^2 for a fixed value of τ^2. Let $F(\sigma^2, \tau^2) = \psi\left(\frac{R^2}{\sigma^2}, 1-\frac{1}{\tau^2}\right)$ represent the impact probability function. Assume, here, without loss of generality, that $R = 1$. The function $F(\sigma^2, \tau^2)$ is strictly decreasing in τ^2 for a fixed σ^2. This is so because the family of negative binomial distributions, with probability functions as in (2.2), is a Monotone Likelihood Ratio (MLR) family. Therefore, by applying a theorem of

Karlin [6], $E_\theta\left\{\text{Pos}\left(J\middle|\frac{1}{2\sigma^2}\right)\right\}$ is an increasing function of θ for each fixed σ^2. Moreover, since $\theta = 1 - \frac{1}{\tau^2}$ and $F(\sigma^2, \tau^2) = 1 - E_\theta\left\{\text{Pos}\left(J \mid \frac{1}{2\sigma^2}\right)\right\}$ we get $F(\sigma^2, \tau^2)$ is a decreasing function of τ^2. Thus, the impact probability assumes its minimal value over C_α along the boundary of the region, $B_\alpha(\tau^2)$. This boundary is given by the formula

$$B_\alpha(\tau^2) = \hat{\sigma}_\alpha^2 + \hat{v}_\alpha^2/\tau^2, \tau^2 \geq 1. \tag{4.2}$$

One would tend to choose the minimal value of $F(\sigma^2, \tau^2)$ along $B_\alpha(\tau^2)$ as a lower confidence limit. However, we can show that the function $F(\sigma^2, \tau^2)$ approaches zero along the boundary, as $\tau^2 \to \infty$. We have therefore investigated by Monte Carlo simulation in section 6 the characteristics of the following three lower limits of interval estimators:

$$\underline{\psi}^{(1)} = F(\hat{\sigma}_\alpha^2 + \hat{v}_\alpha^2, 1), \tag{4.3}$$

$$\underline{\psi}^{(2)} = \sup_{1 \leq \tau < \infty} F(\hat{\sigma}_\alpha^2 + \hat{v}_\alpha^2/\tau^2, \tau^2), \tag{4.4}$$

and

$$\underline{\psi}^{(3)} = F(\hat{\sigma}_\alpha^2 + \hat{v}_\alpha^2/\hat{\tau}^2, \hat{\tau}^2), \tag{4.5}$$

where $\hat{\tau}^2 = \max(1, Q_2/Q_1)$ is the maximum likelihood estimator of τ^2. As shown in Figure 1, $\underline{\psi}^{(1)}$ is the value of the impact probability $F(\sigma^2, \tau^2)$ on the boundary at $\tau^2 = 1$. This lower limit is nearly optimal if the two variances are close. The estimate $\underline{\psi}^{(2)}$ is the maximal value of $F(\sigma^2, \tau^2)$ on the boundary. This estimator is daring in the sense that it provides the shortest interval. The estimate $\underline{\psi}^{(3)}$ is an estimator of the UMA lower limit for the case of known ratio τ^2. Due to the strong consistency of $\hat{\tau}^2$, as an MLE estimator of τ^2, $\underline{\psi}^{(3)}$ is asymptotically a UMA lower limit. In the next section we provide an interval estimator based on the asymptotic distribution of the MLE of the impact probability. We will then demonstrate that $\underline{\psi}^{(3)}$ is best from the point of view of confidence probability coverage and ease of computation.

5. APPROXIMATE LOWER CONFIDENCE LIMITS BASED ON THE ASYMPTOTIC DISTRIBUTION OF MAXIMUM LIKELIHOOD ESTIMATORS

The maximum likelihood estimators (M.L.E.'s) of σ^2 and v^2, based on the sufficient statistics (Q_1, Q_2) are

$$\hat{\sigma}^2 = \frac{Q_1}{n} \text{ and } \hat{v}^2 = \max\left(\hat{\sigma}^2, \frac{Q_2}{n}\right).$$

By applying the invariance principle of maximum likelihood estimation, the M.L.E. of the impac probability $F(\sigma^2, \tau^2)$ is $F(\hat{\sigma}^2, \hat{\tau}^2)$, where $\hat{\tau}^2 = \hat{v}^2/\hat{\sigma}^2$. The family of distribution functions of $(Q_1, Q_2$

for all $o < \sigma^2 < \infty$, $\sigma^2 \leq v^2 < \infty$ and the M.L.E. $F(\hat{\sigma}^2, \hat{\tau}^2)$ satisfy the required regularity conditions for best asymptotically normal estimators.

We shall therefore consider the lower confidence limit based on the asymptotic normal distribution of $F(\hat{\sigma}^2, \hat{\tau}^2)$, ψ_α^*, which is

$$\underline{\psi}_\alpha^* = F(\hat{\sigma}^2, \hat{\tau}^2) - z_{1-\alpha} \text{ S.E. } \{F(\hat{\sigma}^2, \hat{\tau}^2)\}, \tag{5.2}$$

where $z_{1-\alpha}$ is the $(1-\alpha)$ fractile of the standard normal distribution. In (5.2),

$$\text{S.E. } \{F(\hat{\sigma}^2, \hat{\tau}^2)\} = (\text{A.V. } \{F(\hat{\sigma}^2, \hat{\tau}^2)\})^{1/2}, \tag{5.3}$$

where A.V. $\{F(\hat{\sigma}^2, \hat{\tau}^2)\}$ is the asymptotic variance of the M.L.E. evaluated at $(\hat{\sigma}^2, \hat{\tau}^2)$. The asymptotic variance of the M.L.E. of $F(\sigma^2, \tau^2)$ is given by

$$\text{A.V. } F(\hat{\sigma}^2, \hat{\tau}^2) \} = \nabla (I(\sigma^2, \tau^2))^{-1} \nabla \tag{5.4}$$

where ∇ is the gradient function of $F(\sigma^2, \tau^2)$, i.e., $\nabla' = \left(\frac{\partial}{\partial \sigma^2} F(\sigma^2, \tau^2), \frac{\partial}{\partial \tau^2} F(\sigma^2, \tau^2) \right)$ and $I(\sigma^2, \tau^2)$ is the Fisher information matrix.

Let $I_{ij}(\sigma^2, \tau^2)$, $i, j = 1, 2$, denote the (i, j)th element of the Fisher information matrix. These elements are given by

$$I_{ij}(\sigma^2, \tau^2) = E \left\{ \frac{\partial}{\partial \theta_i} \log f(Q_1, Q_2 \mid \sigma^2, \tau^2) \cdot \frac{\partial}{\partial \theta_j} \log f(Q_1, Q_2 \mid \sigma^2, \tau^2) \right\}, \tag{5.5}$$

in which $\theta_1 = \sigma^2$ and $\theta_2 = \tau^2$ and $f(Q_1, Q_2, \mid \sigma^2, \tau^2)$ denotes the joint p.d.f. of (Q_1, Q_2) under σ^2, τ^2. It is a straight-forward matter to demonstrate that

$$I^{-1}(\sigma^2, \tau^2) = \frac{2}{n} \begin{pmatrix} \sigma^4 & -\sigma^2\tau^2 \\ -\sigma^2\tau^2 & 2\tau^4 \end{pmatrix}. \tag{5.6}$$

In the quadratic form (5.4) we can replace $F(\sigma^2, \tau^2)$ by $G(\sigma^2, \tau^2) = 1 - F(\sigma^2, \tau^2)$. From (2.2), and (2.5), we obtain

$$G(\sigma^2, \tau^2) = \sum_{j=0}^{\infty} g\left(j \middle| 1 - \frac{1}{\tau^2}\right) \text{Pos}\left(j \middle| \frac{1}{2\sigma^2}\right). \tag{5.7}$$

Hence,

$$\frac{\partial}{\partial \sigma^2} G(\sigma^2, \tau^2) = \frac{1}{2\sigma^4} \sum_{j=0}^{\infty} g\left(j \middle| 1 - \frac{1}{\tau^2}\right) p\left(j \middle| \frac{1}{2\sigma^2}\right) \tag{5.8}$$

and

$$(5.9)\quad \frac{\partial}{\sigma\tau^2} G(\sigma^2, \tau^2) = -\frac{1}{2\tau^2}\left\{ \sum_{j=0}^{\infty} g^*\left(j \mid 1-\frac{1}{\tau^2}\right) \mathrm{Pos}\left(j \middle| \frac{1}{2\sigma^2}\right) - \sum_{j=0}^{\infty} g^*\left(j \mid 1-\frac{1}{\tau^2}\right) \mathrm{Pos}\left(j+1 \middle| \frac{1}{2\sigma^2}\right)\right\},$$

where

$$(5.10)\quad g^*\left(j \mid 1-\frac{1}{\tau^2}\right) = \frac{1}{\tau^3}\left(\frac{\Gamma\left(j+\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma(j+1)}\right)\left(1-\frac{1}{\tau^2}\right)^j, j=0, 1.$$

The family of negative binomial distributions N.B. (ψ, ν) $0 < \nu < \infty$ (ψ fixed) with probability functions

$$g(j \mid \psi, \nu) = (1-\psi)^{\nu} \frac{\Gamma(j+\nu)}{\Gamma(\nu)\Gamma(j+1)} \psi^j, j = 0, 1, \ldots$$

is an MLR family with respect to j. Therefore, according to Karlin's theorem

$$(5.11)\quad \sum_{j=0}^{\infty} g^*\left(j \mid 1-\frac{1}{\tau^2}\right) \mathrm{Pos}\left(j+1 \middle| \frac{1}{2\sigma^2}\right) > \sum_{j=0}^{\infty} g\left(j \mid 1-\frac{1}{\tau^2}\right) \mathrm{Pos}\left(j \middle| \frac{1}{2\sigma^2}\right)$$

for each σ^2.
Hence, (5.9) is positive and we write,

$$(5.12)\quad \frac{\partial}{\partial\tau^2} G(\sigma^2, \tau^2) = \frac{1}{2\tau^2}\left\{ \sum_{j=0}^{\infty} g^*\left(j \mid 1-\frac{1}{\tau^2}\right) \mathrm{Pos}\left(j+1 \middle| \frac{1}{2\sigma^2}\right) - \sum_{j=0}^{\infty} g\left(j \mid 1-\frac{1}{\tau_2}\right) \mathrm{Pos}\left(j \middle| \frac{1}{2\sigma^2}\right)\right\}.$$

The asymptotic variance is obtained by evaluating (5.4) according to (5.6), (5.8), and (5.12) at $(\hat{\sigma}, \hat{\tau}^2)$. The lower confidence limit is obtained then from (5.2) and (5.3).

6. SIMULATION ESTIMATES OF CONFIDENCE PROBABILITIES AND EXPECTED SHORTAGES

We wish to estimate the average confidence probability, i.e.,

$$(6.1)\quad \mathrm{CP}(\sigma^2, \tau^2) = P_{\sigma^2, \tau^2}\left\{ \underline{\psi}(Q_1, Q_2) \leq \psi\left(\frac{1}{\sigma^2}, 1-\frac{1}{\tau^2}\right)\right\}$$

and the expected shortage

$$(6.2)\quad \mathrm{ES}(\sigma^2, \tau^2) = E_{\sigma^2, \tau^2}\left\{ \left(\psi\left(\frac{1}{\sigma^2}, 1-\frac{1}{\tau^2}\right) - \underline{\psi}(Q_1, Q_2)\right)^+\right\}$$

of the lower limit $\underline{\psi}(Q_1, Q_2)$ where $a^+ = \max(a, 0)$. To accomplish this, we have simulated samples of normal variates (x_i, Y_i), $i = 1, \ldots, n$, $n = 30$, and computed the lower limits $\underline{\psi}(Q_1, Q_2) = \underline{\psi}^{(1)}$, $\underline{\psi}^{(2)}$, $\underline{\psi}^{(3)}$ and $\underline{\psi}^*$ with $\alpha = 0.05$ and 0.025. These are the approximate lower confidence limits suggested in sections 4 and 5. An important question in the design of the simulations is how many independent replicas of the samples under consideration should be taken. We considered this design problem from the point of view of minimizing the expected cost of computation and still achieving good estimates of (6.1) and (6.2).

Since the simulation estimates of the confidence probability are proportions and we wish to attain high precision, we design the simulation from the point of view of testing hypotheses rather than strict point estimation for the following reason: The objective normally would be to estimate the CP with two-decimal-place precision. This requires the standard error of the estimate to be of the order of 10^{-3} and thus results in a prohibitively larger number of replications, i.e., approximately 40,000 replications of samples of size 30 for each trial. On the other hand the testing hypotheses approach requires several hundred replications if we are willing to accept a reasonable deviation between two CP's and a reasonable Type I and Type II Error. Thus in evaluating the CP $= 0.95$ case, we choose $P_1 \leqslant 0.96$ against $P_2 \geqslant 0.975$ with equal error probabilities, $\epsilon = 0.10$. In addition we exploit the Wald Sequential Probability Ratio Test (SPRT).

According to this SPRT, after the i-th replication we compute

$$J = \begin{cases} 1, \text{ if } \underline{\psi} \leqslant \psi \\ 0, \text{ otherwise} \end{cases} \tag{6.3}$$

where $\underline{\psi}$ is the value of the lower limit obtained from the simulated values of (Q_1, Q_2) and ψ is the correct value of the impact probability for (σ^2, τ^2) under consideration. After n observations we compute $S_n = \sum_{i=1}^{n} J_i$ and stop sampling either if $S_n \leqslant -a + bn$ or $S_n \geqslant a + bn$, in which

$$a = \frac{\log \dfrac{1-\epsilon}{\epsilon}}{\log \dfrac{\theta_1}{\theta_0} \cdot \dfrac{1-\theta_0}{1-\theta_1}}.$$

$$b = \frac{\log \dfrac{1-\theta_0}{1-\theta_1}}{\log \dfrac{\theta_1}{\theta_0} \cdot \dfrac{1-\theta_0}{1-\theta_1}}, \tag{6.4}$$

where $\theta_0 = 0.96$ and $\theta_1 = 0.975$. The number of replications was truncated at $N_0 = 400$. The expected number of replications depends on the actual CP value and for the range of (σ^2, τ^2) under consideration varies between 200 and 300. The simulation was performed on a CDC computer at Tel Aviv University, and consumed around 1 minute of cpu-time per combination of (σ^2, τ^2). In Table 1 we provide the simulation estimates of the CP (σ^2, τ^2) and ES (σ^2, τ^2) of $\underline{\psi}^{(1)}$, $\underline{\psi}^{(2)}$ and $\underline{\psi}^{(3)}$ for $\sigma^2 = 1, \tau^2 = 2$ and nominal confidence probability of $P = 0.975$. The exact impact probability is $F(1, 2) = 0.29518$. The actual number of replications in each run is denoted by N.

TABLE 1. *Estimates of CP-values and ES-values for nominal confidence probability* 0.975; $\sigma^2=1$ *and* $\tau^2=2$.

App. Conf. Limit	N	CP	ES
$\underline{\psi}^{(1)}$	400	0.9825	0.096
$\underline{\psi}^{(2)}$	383	0.9765	0.086
$\underline{\psi}^{(3)}$	400	0.9750	0.079

As seen in Table 1, the approximate confidence levels $\underline{\psi}^{(2)}$ and $\underline{\psi}^{(3)}$ yield similar results and both have estimated CP values very close to the nominal confidence probability. It seems that $\underline{\psi}^{(3)}$ is better than $\underline{\psi}^{(1)}$, moreover $\underline{\psi}^{(3)}$ is asymptotically UMA. We therefore concentrate on the approximate confidence limit $\underline{\psi}^{(3)}$.

TABLE 2. *Estimates of the CP and ES Values of* $\underline{\psi}^{(3)}$ *for Various* (σ^2, τ^2) *Combinations with Nominal Confidence Level* $P=0.95$

τ^2		σ^2			
		0.2	0.6	1.	2.
1.	CP	0.925	0.927	0.927	0.925
	ES	0.089	0.113	0.089	0.055
2.	CP	0.943	0.946	0.930	0.930
	ES	0.108	0.090	0.069	0.040
3.	CP	0.931	0.930	0.930	0.927
	ES	0.109	0.081	0.058	0.033
4.	CP	0.931	0.930	0.930	0.927
	ES	0.106	0.072	0.051	0.029
5.	CP	0.931	0.930	0.927	0.927
	ES	0.101	0.066	0.046	0.026
6.	CP	0.931	0.930	0.927	0.927
	ES	0.097	0.061	0.042	0.024
7.	CP	0.931	0.930	0.927	0.927
	ES	0.093	0.057	0.039	0.022
8.	CP	0.931	0.927	0.927	0.927
	ES	0.089	0.053	0.037	0.021
9.	CP	0.931	0.927	0.927	0.927
	ES	0.085	0.051	0.035	0.019
10.	CP	0.930	0.925	0.927	0.927
	ES	0.083	0.049	0.033	0.018

In order to compare $\underline{\psi}^{(3)}$ with the asymptotic UMA confidence limit $\underline{\psi}^*$, given in section 5, we simulated a run, for $\sigma^2 = 1$, $\tau^2 = 2$, in which the limits $\underline{\psi}^*$ were computed for a nominal confidence probabil

ty of P = 0.95. Sampling terminated after N = 193 replications with an estimated CP of 0.93264 and estimated ES of 0.06850. As seen in Table 2, the estimated CP value of $\underline{\psi}^{(3)}$ for $\sigma^2 = 1, \tau^2 = 2$ and nominal confidence probability of P = 0.95 is CP = 0.92982 with N = 171. These two CP values are not significantly different. The lower limit $\underline{\psi}^{(3)}$ is simpler for computation than $\underline{\psi}^*$. *We therefore recommend the use of* $\underline{\psi}^{(3)}$ *as an approximate lower confidence limit for* $F(\sigma^2, \tau^2)$ *when* τ^2 *is unknown.* In Table 2 we provide the estimated CP values and some ES values of $\underline{\psi}^{(3)}$ for various σ^2 and τ^2 combinations.

We see in Table 2 stable results and somewhat regular behavior of the ES function. As σ^2 and τ^2 increase, ES decreases. This is due to the fact that as σ^2 and τ^2 grow the impact probability ψ decreases. When ψ is close to zero the shortage of $\underline{\psi}^{(3)}$ cannot be large. In order to combine the estimates of CP and ES more effectively we combine them in the following measure of relative efficiency R.E.:

$$\text{R.E.} = (\text{CP}/1-\alpha)e^{-\text{ES}}.$$

A procedure is efficient if its confidence probability, CP, is high relative to $(1-\alpha)$. To avoid the possibility that the trivial lower limit $\underline{\psi} \equiv 0$ will be most efficient (since its CP = 1) we multiply by e^{-ES}. Accordingly, the values in Table 2 yield the following R.E.'s listed in Table 3.

TABLE 3. R.E. *of* $\underline{\psi}^{(3)}$

τ^2	σ^2			
	0.2	0.6	1.0	2.0
1	0.891	0.872	0.893	0.921
2	0.891	0.911	0.914	0.941
3	0.879	0.903	0.924	0.944
4	0.879	0.911	0.931	0.948
5	0.886	0.917	0.932	0.951
6	0.889	0.921	0.936	0.953
7	0.893	0.924	0.939	0.955
8	0.897	0.925	0.940	0.956
9	0.900	0.927	0.942	0.958
10	0.901	0.927	0.944	0.958

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ON ESTIMATING THE RELIABILITY OF A COMPONENT SUBJECT TO SEVERAL DIFFERENT STRESSES (STRENGTHS)*

Satish Chandra

University of Texas at El Paso

Donald B. Owen

Southern Methodist University

ABSTRACT

This paper is concerned with estimating $p=P(X_1<Y, \ldots, X_N<Y)$ or $q=P(X<Y_1, \ldots, X<Y_N)$ where the X's and Y's are all independent random variables. Applications to estimation of the reliability p from stress-strength relationships are considered where a component is subject to several stresses $X_1, X_2, \ldots, X_N$ whereas its strength, Y, is a single random variable. Similarly, the reliability q is of interest where a component is made of several parts all with their individual strengths $Y_1, Y_2, \ldots, Y_N$ and a single stress X is applied to the component. When the X's and Y's are independent and normal, maximum likelihood estimates of p and q have been obtained. For the case $N=2$ and in some special cases, minimum variance unbiased estimates have been given. When the Y's are all exponential and the X is normal with known variance, but unknown mean (or uniform between 0 and θ, θ being unknown) the minimum variance unbiased estimate of q is established in this paper.

. INTRODUCTION

In the past few years, many authors have attempted to estimate and give confidence bounds for ıe reliability of a component using probability arguments of a certain physical model of failure. ccording to this model a component fails if at any moment the applied stress (or load), say X, is eater than the resistance, say Y. That is, the problem here is to find an estimate of the probability ıat Y is less than X, where X and Y are both random variables, having some known or unknown proba- lity structure, and to find confidence limits on this probability. An extensive amount of work has ɔen done on this problem by many authors, and some, for example, Mazumdar [7], Church and arris [2], and others have given practical uses for the results.

In this paper the following probability is considered:

.1) $$p=P\{X_1<Y, X_2<Y, \ldots, X_N<Y\}.$$

n example of a problem where p would be of interest is the case where a beam with strength Y is ıbjected to several stresses $X_1, X_2, \ldots, X_N$. In this case p could logically be defined to be the liability of the beam provided the beam fails if and only if one of the X's exceeds the strength Y.

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Another similar problem of interest is to evaluate

$$q = P\{X < Y_1, X < Y_2, \ldots, X < Y_N\}$$
$$= P\{-Y_1 < -X, -Y_2 < -X, \ldots, -Y_N < -X\},$$

where X, Y_1, Y_2, . . ., Y_N are all random variables. Note that $p + q \neq 1$, and that two distinct but related problems are discussed here. An example of a problem where q would be of interest is a component assembled from N items with strengths $Y_1, Y_2, \ldots, Y_N$ and a single stress, X, is applied to all of the components. As long as the stress is less than the individual strengths of each component, the component operates satisfactorily and q can be defined to be the reliability of the component. For example, the flow of a current X through an electonic component assembled from several subcomponents with abilities to accommodate the current $Y_1, Y_2, \ldots, Y_N$, would follow this pattern.

Our principal objective is to estimate p and q defined above in the particular case when the X's and Y's are all independent. The problems of estimating p and q are not symmetrical and hence in some cases below we have estimates for p, but not for q and vice versa.

An expression for p is given in the following lemma:

LEMMA 1.1: Let $T = \max X_i$, $1 \leq i \leq N$ and $P\{X_i \leq x\} = F_i(x)$, $i = 1, 2, \ldots, N$. Then clearly $p = P\{T < Y\}$ and $H(t) = P\{T \leq t\} = \prod_{i=1}^{N} F_i(t)$. If $F_i = F$, $i = 1, 2, \ldots, N$, then $p = \int_{-\infty}^{\infty} [F(y)]^N dG(y)$, where $G(y) = P\{Y \leq y\}$ and if $F = G$, then $p = (N+1)^{-1}$.

Similarly, an expression for q is $q = \int_{-\infty}^{\infty} [1 - G(x)]^N dF(x)$, where $P\{X \leq x\} = F(x)$ and $P\{Y_i \leq y\} = G(y)$ for all i, and if $F = G$, $q = (N+1)^{-1}$.

Sections 2 and 3 below are concerned with estimation of p in the case where the X's and Y's are normal. Section 4 below deals with estimation of q when X is normal and the Y's are exponential. In section 5, we consider the case where we estimate q with X uniform and the Y's are exponential.

2. ESTIMATION OF P, WHEN $X_1, \ldots, X_N, Y$ ARE ALL INDEPENDENT NORMAL

Let Φ and ψ denote the c.d.f. and the density function of a standard normal random variable, respectively. Let $X_1, X_2, \ldots, X_N, Y$ be independent normal with $EX_i = \mu$, $\sigma_{Xi}^2 = \sigma^2$, $EY = \nu$, $\sigma_Y^2 = \tau^2$. Also let $\rho = \dfrac{\tau^2}{\sigma^2 + \tau^2}$ and $K = \dfrac{(\nu - \mu)}{(\sigma^2 + \tau^2)^{1/2}}$, then by Lemma 1.1,

p $\quad$ ϕ^N $\quad$ $\gamma + \tau z - \mu$ $\quad$ ψ $\quad$ zdz

$$p = \int_{-\infty}^{\infty} \Phi^N\left(\frac{\nu + \tau z - \mu}{\sigma}\right) \psi(z)dz$$

$$= \int_{-\infty}^{\infty} \Phi^N\left(\frac{\rho^{1/2} z + K}{(1-\rho)^{1/2}}\right) \psi(z)dz$$

$$= F_N(K; \rho), \text{ say.}$$

The function $F_N(K, \rho)$ is the probability that each of N standardized normal variables with equal correlation ρ and having a joint normal distribution will not exceed K. For this particular case the function $F_N(K, \rho)$ has been tabulated by Shanti S. Gupta [4] for given values of $\rho(\rho \geq 0)$, N and K.

Thus if μ, ν, σ^2, τ^2 are known we can evaluate p using Gupta's table. However, if μ, ν, σ^2, τ^2 are unknown or some of them are unknown, we can use their maximum likelihood estimators. If μ, ν, σ^2 and τ^2 are unknown, and if $\hat{\mu}$, $\hat{\nu}$, $\hat{\sigma}^2$ and $\hat{\tau}^2$ be their maximum likelihood estimators respectively, then using Zehna's [8] generalization of the notion of maximum likelihood estimators (which states that the maximum likelihood estimator of a function of parameters is the same function of the maximum likelihood estimator of the parameters), the maximum likelihood estimator of p is

$$\hat{p} = F_N(\hat{K}, \hat{\rho}),$$

where

$$\hat{\rho} = \frac{\hat{\tau}^2}{\hat{\sigma}^2 + \hat{\tau}^2}, \qquad \hat{K} = \frac{\hat{\nu} - \hat{\mu}}{(\hat{\sigma}^2 + \hat{\tau}^2)^{1/2}}.$$

This estimator, again, can be evaluated using Gupta's table [4] for $N = 1(1)12; K = -3.5(0.1)3.5$; and $\rho = 0.1, 0.125, 0.2, 0.25, 0.3, 1/3, 0.375, 0.4, 0.5, 0.6, 0.625, 2/3, 0.7, 0.75, 0.8, 0.875, 0.9$.

3. MINIMUM VARIANCE UNBIASED ESTIMATOR OF p IN SOME CASES

Let $n_k(x_1, x_2, \ldots, x_k; \underline{\mu}, \Sigma)$ and $N_k(x_1, x_2, \ldots, x_k; \underline{\mu}, \Sigma)$ denote the k-variate normal density function and distribution function, respectively, with mean vector μ and covariance matrix $\Sigma = (\sigma_{ij})$. Further, let X_1, X_2, Y be independent normal with means μ_1, μ_2 and μ_y, respectively, and equal iance σ^2. Then we want to estimate

$$p = P\{X_1 < Y, X_2 < Y\}$$
$$= P\{X_1 - Y < 0, X_2 - Y < 0\}.$$

The distribution of $(X_1 - Y, X_2 - Y)$ is $n_2(x, y; (\mu_1 - \mu_y, \mu_2 - \mu_y), \sigma_{ii} = 2\sigma^2, \sigma_{ij} = \sigma^2, i \neq j)$. Then $p = N_2(2^{-1/2}\sigma^{-1}(\mu_y - \mu_1), 2^{-1/2}\sigma^{-1}(\mu_y - \mu_2); \underline{0}, \sigma_{ii} = 1, \sigma_{ij} = 1/2, i \neq j)$, where $\underline{0}$ is the zero vector.

THEOREM 3.1: If the mean μ_y and variance σ^2 are known, but the means μ_1 and μ_2 are unknown, and data $(x_{11}, x_{12} - \ldots, x_{1l})$, $(x_{21}, x_{22}, \ldots, x_{2m})$ on X_1 and X_2, respectively, are available, then the minimum variance unbiased (m.v.u.) estimator of p is given by

$$\hat{p} = N_2\left\{\left(2 - \frac{1}{l}\right)^{-1/2}\sigma^{-1}(\mu_y - \bar{x}_1), \left(2 - \frac{1}{m}\right)^{-1/2}\sigma^{-1}(\mu_y - \bar{x}_2), \underline{0}, \sigma_{ii} = 1, \right.$$
$$\left. \sigma_{ij} = \left(2 - \frac{1}{l}\right)^{-1/2}\left(2 - \frac{1}{m}\right)^{-1/2}, i \neq j\right\}, \tag{3.1}$$

where

$$\bar{x}_1 = \frac{1}{l}\sum_{i=1}^{l} x_{1i}, \text{ and } \bar{x}_2 = \frac{1}{m}\sum_{j=1}^{m} x_{2j}.$$

Moreover, as l and m become large, this estimator asymptotically approaches the maximum likelihood estimator.

PROOF: Let W be a normal random variable, independent of X_1 and X_2, with mean 0 and variance σ^2, and let

$$I = \begin{cases} 1 & \text{if } x_{11} + W < \mu_y,\ x_{21} + W < \mu_y \\ 0 & \text{otherwise.} \end{cases}$$

Then $E(I) = p$. Since $(\bar{x}_1, \bar{x}_2)$ is a complete sufficient statistic, it follows from the Rao-Blackwell and Lehmann-Scheffé theorems that the conditional expectation $E(I|\bar{x}_1, \bar{x}_2)$ is the m.v.u. estimator of p. Now the distribution of $(\bar{x}_1, \bar{x}_2, x_{11}+W, x_{21}+W)$ is $n_4(t_1, t_2, t_3, t_4; (\mu_x, \mu_y, \mu_x, \mu_y), \Sigma)$, where

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{11} \\ \Sigma_{11} & \Sigma_{22} \end{pmatrix},\ \Sigma_{11} = \begin{pmatrix} \frac{\sigma^2}{l} & 0 \\ 0 & \frac{\sigma^2}{m} \end{pmatrix} \text{ and } \Sigma_{22} = \begin{pmatrix} 2\sigma^2 & \sigma^2 \\ \sigma^2 & 2\sigma^2 \end{pmatrix}$$

Furthermore, the conditional distribution of $(x_{11}+W, x_{21}+W)$ given $(\bar{x}_1, \bar{x}_2)$ is $n_2\left(t_1, t_2; (\bar{x}_1, \bar{x}_2), \sigma_{11} = 2\sigma^2 - \frac{\sigma^2}{l}, \sigma_{22} = 2\sigma^2 - \frac{\sigma^2}{m}, \sigma_{ij} = \sigma^2, i \neq j\right)$.

Hence the m.v.u. estimator of p is

$$\hat{p} = P\{x_{11}+W < \mu_y,\ x_{21}+W < \mu_y \mid \bar{x}_1, \bar{x}_2\},$$

which, after simplification, reduces to (3.1).

THEOREM 3.2: If the means μ_1, μ_2 and variance σ^2 are known, but μ_y is unknown and data $(y_1, \ldots, y_r)$ on Y are available, then the m.v.u. estimator of p is given as

$$\hat{p} = N_2\left\{\left(2-\frac{1}{r}\right)^{-1/2}\sigma^{-1}(\bar{y}-\mu_1), \left(2-\frac{1}{r}\right)^{-1/2}\sigma^{-1}(\bar{y}-\mu_2); \underline{0}, \sigma_{ii}=1, \sigma_{ij}=\frac{r-1}{2r-1}, i \neq j\right\},$$

where

$$\bar{y} = \frac{1}{r}\sum_{k=1}^{r} y_k.$$

PROOF: The proof is similar to the proof of Theorem 3.1.

4. ESTIMATION OF q WHEN X HAS A NORMAL DISTRIBUTION WITH MEAN μ, VARIANCE σ^2 AND $Y_1, \ldots, Y_N$ ARE ALL INDEPENDENT EXPONENTIAL

Let $Y_1, \ldots, Y_N$ all be independent and identically distributed with the density function

$$g(y) = \begin{cases} \alpha\, e^{-\alpha y},\ y > 0,\ \alpha > 0 \\ 0 \qquad \text{otherwise,} \end{cases} \tag{4.1}$$

and the distribution function

$$G(y)=\begin{cases}0 & \text{if } y\leq 0\\ 1-e^{-\alpha y} & \text{if } y>0.\end{cases} \tag{4.2}$$

Next, suppose that X is a normal variable with mean μ and variance σ^2. We are interested in estimating $q=P\{X<Y_1, X<Y_2, \ldots, X<Y_N\}$.

In this situation, we have the following result.

LEMMA 4.1: If X is normally distributed with mean μ and variance σ^2 and $Y_1, \ldots, Y_N$ are independent identically distributed with density function (4.1), then

$$q=\Phi(-\mu/\sigma)+e^{-\alpha N\mu+\alpha^2N^2\sigma^2/2}\left[1-\Phi\left(\alpha N\sigma-\frac{\mu}{\sigma}\right)\right]. \tag{4.3}$$

PROOF: The proof follows from the remark about q given after Lemma 1.1. Now, suppose that a random sample distributed as X, say $x_1, \ldots, x_m$ is observed. The problem here is to find the minimum variance unbiased estimator of q by the use of this sample.

THEOREM 4.1: If σ and α are known and $m>2$, the minimum variance unbiased estimator of q is

$$\hat{q}=\hat{q}_1+\hat{q}_2, \tag{4.4}$$

where

$$\hat{q}_1=\Phi\left(-\sqrt{m}\,\bar{x}/\sigma\sqrt{m-1}\right),$$

and

$$\hat{q}_2=e^{-\alpha N\bar{x}+\alpha^2N^2(m-1)\sigma^2/2m}\cdot\left[1-\Phi\left\{\frac{(m-1)^{\frac{3}{2}}\alpha N\sigma^2-m\sqrt{m-1}\,\bar{x}}{\sigma\sqrt{m(1+m(m-2))}}\right\}\right],$$

and as m becomes large, this estimator is asymptotically equal to the maximum likelihood estimator of q.

PROOF: Let $q=q_1+q_2$ where

$$q_1=\Phi(-\mu/\sigma)$$

and

$$q_2=e^{-\alpha N\mu+\alpha^2N^2\sigma^2/2}\left[1-\Phi\left(\alpha N\sigma-\frac{\mu}{\sigma}\right)\right].$$

To find the m.v.u. estimator of q, it would be enough to find an unbiased estimator of q which is a function of a complete sufficient statistic, in view of the Rao-Blackwell and Lehmann-Scheffé the-

orems. A complete sufficient statistic in this case is $\bar{x}$ and if we can find an unbiased estimator $\hat{q}_1$ of q_1 and an unbiased estimator $\hat{q}_2$ of q_2 which are functions of a complete sufficient statistic, then the m.v.u. estimator of q will be

$$\hat{q} = \hat{q}_1 + \hat{q}_2.$$

Following Lieberman and Resnikoff [5], the m.v.u. estimator of q_1 is given by

$$\hat{q}_1 = \Phi(-\sqrt{m}\, x/\sigma\sqrt{m-1}),$$

which is a function of a complete sufficient statistic.

Thus, we only need to find an unbiased estimator of q_2 which is a function of a complete sufficient statistic.

Let $t_1 = e^{-\alpha N x}$ 1, then $E(t_1) = e^{-\alpha N\mu + \alpha^2 N^2\sigma^2/2}$. Further, let

$$t_2 = \begin{cases} 0 & \text{if } -\infty < x_2 < \alpha N\sigma^2 \\ 1 & \text{otherwise,} \end{cases}$$

then

$$E(t_2) = 1 - \text{Prob}\,(-\infty < x_2 < \alpha N\sigma^2) = 1 - \Phi\left(\alpha N\sigma - \frac{\mu}{\sigma}\right).$$

Clearly, t_1 and t_2 are independent and if $t = t_1 t_2$, then $E(t) = E(t_1)E(t_2) = q_2$. Thus, in view of Rao-Blackwell, Lehmann-Scheffé theorems, since $\bar{x}$ is a complete sufficient statistic, $E(t \mid \bar{x})$ is the m.v.u. estimator of q_2. Now, the distribution of $(\bar{x}, x_1, x_2)$ is $n_3(x_1, x_2, x_3;\ (\mu, \mu, \mu),\ \sigma_{il} = \sigma_{li} = \dfrac{\sigma^2}{m},\ \sigma_{jj} = \sigma^2,$ $j = 2, 3,\ \sigma_{23} = \sigma_{32} = 0)$, and the conditional distribution of (x_1, x_2) given $\bar{x}$ is $n_2\,(x_1, x_2;\ (\bar{x}, \bar{x}),\ \sigma_{ii} = \sigma^2 - \sigma^2/m,\ \sigma_{ij} = -\sigma^2/m,\ i \neq j)$. Thus the conditional expectation, after simplification reduces to

$$E(t/\bar{x}) = e^{-\alpha N\bar{x} + \alpha^2 N^2 (m-1)\sigma^2/2m} \cdot \left[1 - \int_{-\infty}^{\infty} \frac{\sqrt{m}}{\sqrt{m-1}\,\sqrt{2\pi}\,\sigma} \Phi\left\{\frac{\alpha N \cdot \sqrt{(m-1)}}{\sqrt{m-2}}\sigma\right.\right.$$

$$\left.\left. - \frac{m}{\sqrt{(m-1)(m-2)}\sigma}\left(\bar{x} - \frac{x_1}{m}\right)\right\} \exp\left[-\frac{m}{2(m-1)\sigma^2}\left\{x_1 - \left(\bar{x} - \frac{\alpha N(m-1)\sigma^2}{m}\right)\right\}^2\right] dx_1\right].$$

To evaluate the integral, use the transformation

$$t = \frac{x_1 - \left(\bar{x} - \dfrac{\alpha N(m-1)\sigma^2}{m}\right)}{\sqrt{(m-1)/m}\ \sigma},$$

and let

$$a = \frac{(m-1)^{3/2}}{m \cdot \sqrt{(m-2)}}\alpha N\sigma - \frac{\sqrt{m-1}}{\sqrt{m-2}} \cdot \frac{\bar{x}}{\sigma},$$

$$b=\frac{1}{\sqrt{m(m-2)}},$$

then the integral becomes

$$\int_{-\infty}^{\infty}\psi(t)\Phi(a+bt)\,dt=\Phi\left(\frac{a}{\sqrt{1+b^2}}\right).$$

Therefore

$$\hat{q}_2=E(t|\bar{x})=e^{-\alpha N\bar{x}+\alpha^2N^2(m-1)\sigma^2/2m}\left[1-\Phi\left\{\frac{(m-1)^{3/2}\alpha N\sigma^2-m(m-1)^{1/2}\bar{x}}{\sigma(m)^{1/2}[m(m-2)+1]^{1/2}}\right\}\right],$$

and the minimum variance unbiased estimator of q is therefore given by (4.4). The maximum likelihood estimator (in the sense of Zehna [8] of q when σ and α are known is given as

$$\hat{q}=\Phi\left(-\frac{\bar{x}}{\sigma}\right)+e^{-\alpha N\bar{x}+\alpha^2N^2\sigma^2/2}\left[1-\Phi\left(\alpha N\sigma-\frac{\bar{x}}{\sigma}\right)\right],$$

and thus as m becomes large, the minimum variance unbiased estimator and the maximum likelihood estimator are asymptotically equivalent.

Note that if μ, σ and α are all unknown in Theorem 4.1, then the maximum likelihood estimator (in the sense of Zehna [8]) can be obtained by substituting the corresponding maximum likelihood estimators of the parameters in the functional form.

5. ESTIMATION OF q WHEN X IS UNIFORM AND $Y_1, \ldots, Y_N$ ARE ALL INDEPENDENT EXPONENTIAL.

Let $Y_1, \ldots, Y_N$ be all independent and identically distributed with the density function and distribution function given by (4.1) and (4.2), respectively, and X be a uniform random variable between 0 and θ, i.e., the density function of X is given by

$$f(x)=\begin{cases}\frac{1}{\theta} & \text{if } 0<x<\theta\\ 0 & \text{otherwise.}\end{cases}$$

Then we have the following result.

THEOREM 5.1: In this case $q=(1-e^{-\alpha N\theta})/\alpha N\theta$ and the m.v.u. estimator of q, when α is known, is

$$\hat{q}=\frac{m-1}{\alpha mNt}+\frac{1}{m}\,e^{-\alpha Nt}\left(1-\frac{m-1}{\alpha Nt}\right),$$

where $t = \max(x_1, \ldots, x_m)$ and $x_1, \ldots, x_m$ is an observed random sample distributed as X.

PROOF: Let $u = e^{-\alpha N x_1}$, then $E(u) = q$. Since t is a complete sufficient statistic, $E(u \mid t)$ is m.v.u. estimator of q, and the result follows.

As an example of the use of this section consider electric current, X, supplied to a component consisting of several different transistors, each having different abilities to accommodate the current. Also, if the abilities, $Y_1, Y_2, \ldots, Y_N$, of different transistors to accommodate current follow a random phenomenon during a given interval $[0, T]$, then the component does not fail as long as applied force. or stress, X, is simultaneously less than the several different abilities to handle the stress during the given interval. In this case, the Y's can be considered to be exponentially distributed with a suitable parameter and X to be uniformly distributed in the interval $(0, \theta)$. We then compute our estimate of q using Theorem 5.1.

Another model for this situation is to assume the X to be approximately normally distributed (with the Y's still exponential) and then we would use Theorem 4.1 to obtain our estimate.

6. CONCLUSION

In a very few cases, it was possible to derive the minimum variance unbiased estimators, whereas in other cases the derivation of the minimum variance unbiased estimators remain an open question. For example, in Theorem 4.1, if σ is also unknown, which is the more realistic case, we could not derive the minimum variance unbiased estimate. The problem lies in finding the m.v.u. estimator of $\Phi(c\sigma^2)$, where c is a constant, and Φ is the cumulative distribution function of a normal variate with mean μ, and variance σ^2. However, Gray, Watkins and Schucany [3] have recently given a representation of such an estimator. In any event, maximum likelihood estimators have been given here.

As an alternative to overcome such difficulties, a nonparametric method of estimating p and q has been considered by Chandra [1], where only the continuity of the cumulative distribution functions of the $N+1$ random variables is assumed, and a statistic W, based on ranks of the pooled sample values from all the random variables, has been suggested and shown to have certain optimum properties, i.e., it is an unbiased and consistent estimator.

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PROPERTIES OF AN APPROXIMATE HAZARD TRANSFORM*

J. D. Esary

and

W. J. Hayne

Naval Postgraduate School Monterey, California

ABSTRACT

The calculation of the exact reliability of complex systems is a difficult and tedious task. Consequently simple approximating techniques have great practical value.

The hazard transform of a system is an invertible transformation of its reliability function which is convenient and useful in both applied and theoretical reliability work. A simple calculus for finding an approximate hazard transform for systems formed by series and parallel combinations of components is extended so that it can be used for any coherent system. The extended calculus is shown to lead to conservative approximations.

A first order version of the extended calculus is also discussed. This method of approximation is even more simple to use, but is not always conservative. Examples of its application indicate that it is capable of giving quite accurate results.

1. INTRODUCTION

Suppose that two components perform independently and have probabilities p_1 and p_2 of doing so successfully, i.e., of *functioning*. If the components constitute a *series* system, i.e., a system that functions only if both of its components function, then the probability that the system functions is

$$p = h(p_1, p_2) = p_1 p_2,$$

where $h(p_1, p_2)$ is a *reliability function*. With due regard for the limitations imposed by assuming that the components perform independently, the reliability function of a two component series system is a convenient summary of its stochastic properties. The convenience stems from the variety of interpretations that can be attached to the term functioning, for the components, and consequently for the system. For example, p_1 and p_2 could be component success probabilities for a mission that develops in time, or one of several types of component availabilities. In each case p, as computed from $h(p_1, p_2)$, will be the corresponding quantity for the system. The reliability functions of more complex systems have the same utility.

The quantities $u_1 = -\log p_1$, $u_2 = -\log p_2$ are the component *hazards*, and $u = -\log p$ is the system hazard. For the two component series system

$$u = \eta(u_1, u_2) = u_1 + u_2,$$

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where $\eta(u_1, u_2)$ is a *hazard transform.* The hazard transform of a system is equivalent to its reliability function, but for many purposes represents its properties in an even more convenient way. Examples of the application of hazard transforms are given in Esary, Marshall, and Proschan (1970).

For the two component *parallel* system, i.e., the system that functions if either of its components functions, the reliability function is

$$h(p_1, p_2) = 1 - (1-p_1)(1-p_2),$$

and the hazard transform is

$$\eta(u_1, u_2) = -log\ \{1 - (1-e^{-u_1})(1-e^{-u_2})\}.$$

The contrast in complexity between the series and parallel hazard transforms is apparent. However, for small component hazards u_1, u_2, i.e., for large component success probabilities p_1, p_2,

$$\eta(u_1, u_2) \simeq u_1 u_2$$

is a good approximation for the parallel system, and is conservative in that it overestimates system hazard and thus underestimates system reliability.

An *approximate hazard transform* η^* can be defined by:

$$\eta^*(u_1, u_2) = \eta(u_1, u_2) = u_1 + u_2 \text{ for two components in series,}$$

$$\eta^*(u_1, u_2) = u_1 u_2 \text{ for two components in parallel.}$$

These definitions lead to a simple calculus for finding an approximate hazard transform for systems that can be formed using series and parallel combinations of components. For example, the system with the

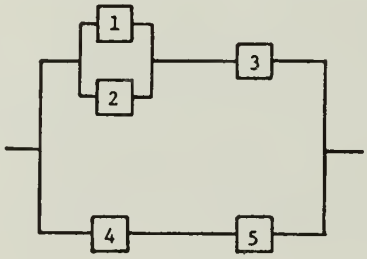

FIGURE 1.

reliability block diagram shown in Figure 1 has, following the above rules of computation, the approximate hazard transform

$$\eta^*(u_1, \ldots, u_5) = (u_1 u_2 + u_3)(u_4 + u_5). \tag{1.1}$$

Rubinstein [6] introduced this calculus as a step in deriving life test procedures for large systems,

ınd considered (Ref. [7]) more refined approximations for parallel and some other systems. The calculus s employed in the Guide Manual for Reliability Measurement [8] issued by the Navy Special Projects)ffice.

Our purpose is to note a simple extension of the calculus to systems that cannot be formed using ust series and parallel combinations of components, e.g., the system with the block diagram shown in 'igure 2, *and to show that the extended calculus is conservative.*

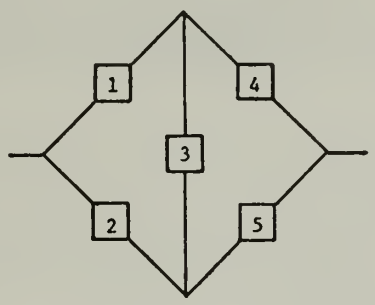

FIGURE 2.

We also comment on a first order version of the extended calculus which is not necessarily conervative, but which gains in simplicity and can give quite accurate results.

2. AN APPROXIMATE HAZARD TRANSFORM

The approximate hazard transform we consider can be defined for the class of coherent systems. ystems describable by a reliability block diagram or by a fault tree using "and" and "or" gates are oherent. Alternately, the performance of the components in a system can be indicated by Bernoulli andom variables $X_1, \ldots, X_n$, where $X_i = 1$ if the ith component functions, $X_i = 0$ if the ith component ails to function, and the performance of the system can be indicated by a *structure function* $\phi(\mathbf{X}) = \phi(X_1, \ldots, X_n)$, where $\phi(\mathbf{X}) = 1$ if the system functions, $\phi(\mathbf{X}) = 0$ if the system fails to function. The system is *coherent* if ϕ is increasing in each of its coordinates and $\phi(1, \ldots, 1) = 1$, $\phi(0, \ldots, 0) = 0$, conditions which are clearly satisfied by systems described by block diagrams or ault trees.

If the components in a system perform independently, i.e., if $X_1, \ldots, X_n$ are independent, hen the probability $p = P[\phi(\mathbf{X}) = 1]$ that the system functions can be computed from the marginal robabilities $p_i = P[X_i = 1]$, $i = 1, \ldots, n$, that the components function. If not, then p depends on he joint distribution of $X_1, \ldots, X_n$. The *reliability function* h of a system describes the relationhip between p and $p_1, \ldots, p_n$ in the case of independence. It is formally defined by

$$h(\mathbf{p}) = h(p_1, \ldots, p_n) = P[\phi(\mathbf{X}) = 1],$$

where $X_1, \ldots, X_n$ are independent and $P[X_i = 1] = p_i$, $0 \leq p_i \leq 1$, $i = 1, \ldots, n$.

It is common in the assurance disciplines to work with hazards rather than with probabilities. Examples are the "parts count" method in which component hazards are added to obtain a system ıazard (in effect assuming that the system is series), and the practice of adding hazards over phases of a mission to obtain a mission hazard.

Recall that the component *hazards* are $u_i = -\log p_i$, so that $p_i = e^{-u_i}$, $i = 1, \ldots, n$, and the

system *hazard* is $u = -\log p = -\log h(\mathbf{p})$. A hazard is zero when the probability of functioning is one and increases to infinity as the probability of functioning decreases to zero, which to some extent makes the name appropriate. The *hazard transform* η of the system relates u to $u_1, \ldots, u_n$ and is defined by

$$\eta(\mathbf{u}) = \eta(u_1, \ldots, u_n) = -\log h(\mathbf{p}) = -\log h(e^{-u_i}, \ldots, e^{-u_n}),$$

where $u_i \geq 0$, $i = 1, \ldots, n$. Knowing the hazard transform of a system is equivalent to knowing its reliability function since

$$h(\mathbf{p}) = e^{(n)\mathbf{u}-} = e^{-\eta(-\log p_1, \ldots, -\log p_n)},$$

The assumption that components perform independently is implicit in the definition of a hazard transform, just as it is in the definition of a reliability function.

The approximate hazard transform we consider can be conveniently introduced by first considering an approximation for the hazard transform of a parallel system and then extending the approximation to an arbitrary coherent system by using a representation of the system as a series of parallel subsystems related to its "minimal cuts."

The n component *parallel* system functions if at least one of its components functions, so that its structure function is

$$\phi(\mathbf{X}) = 1 - \prod_{i=1}^{n} (1 - X_i),$$

and its reliability function is

$$h(\mathbf{p}) = P[\phi(\mathbf{X}) = 1] = 1 - P[X_1 = 0, \ldots, X_n = 0]$$

$$= 1 - \prod_{i=1}^{n} P[X_i = 0] = 1 - \prod_{i=1}^{n} (1 - p_i).$$

Its hazard transform is

$$\eta(\mathbf{u}) = -\log h(e^{-u_1}, \ldots, e^{-u_n}) = -\log \left[1 - \prod_{i=1}^{n} (1 - e^{-u_i})\right]. \tag{2.1}$$

An *approximate hazard transform* for the n component parallel system is

$$\eta^*(\mathbf{u}) = \prod_{i=1}^{n} u_i, \qquad \mathbf{u} \geq \mathbf{0}, \tag{2.2}$$

where $\mathbf{0}=(0, \ldots, 0)$ and $\mathbf{u} \geq \mathbf{0}$ means $u_i \geq 0$, $i=1, \ldots, n$. The approximation is based on the power series expansion for $\eta(\mathbf{u})$.

The following lemma shows, for parallel systems, that: (a) the exact and approximate hazard transforms agree for perfectly reliable components, (b) the approximate hazard transform is conservative, i.e., it indicates greater hazard (less reliability) that the exact transform, and (c) the accuracy of the approximate hazard transform decreases as the component hazards increase. The lemma extends an observation of Rubinstein [7, appendix B].

LEMMA 2.1: *For an* n *component parallel system the following comparisons exist between the hazard transform* η *and the approximate hazard transform* η^*:

(a) $\eta^*(\mathbf{0})=\eta(\mathbf{0})=0$.

(b) $\eta^*(\mathbf{u}) \geq \eta(\mathbf{u})$, $\mathbf{u} \geq \mathbf{0}$.

(c) $\eta^*(\mathbf{u})-\eta(\mathbf{u})$ *is increasing with respect to* u_i, $i=1, \ldots, n$.

PROOF: It is immediate that $\eta^*(\mathbf{0})=0$ and that $\eta(\mathbf{0})=-\log 1=0$. Thus (a) holds. That (b) holds follows from (a) and (c). To verify (c) note that

$$\frac{\partial}{\partial u_i}\{\eta^*(\mathbf{u})-\eta(\mathbf{u})\}=\prod_{j\neq i} u_i-\frac{e^{-u_i}\prod_{j\neq i}(1-e^{-u_j})}{1-\prod_{j=1}^{n}(1-e^{-u_j})}.$$

Since $u_j \geq 1-e^{-u_j}$ so that $\prod_{j\neq i} u_j \geq \prod_{j\neq i}(1-e^{-u_j})$, and $1-\prod_{j=1}^{n}(1-e^{-u_j}) \geq e^{-u_i}$ (the reliability of a parallel system is not less than the reliability of one of its components), it follows that $\frac{\partial}{\partial u_i}\{\eta^*(\mathbf{u})-\eta(\mathbf{u})\} \geq 0$. Thus (c) holds. □

The assumption that the components perform independently is crucial to the comparisons of Lemma 2.1, as is shown by the following example.

EXAMPLE 2.2: Suppose the components in a two component parallel system perform dependently in the strong, positive sense that $X_1=X_2=X$, i.e., if one component fails, so does the other. Then $\phi(X_1, X_2)=1-(1-X_1)(1-X_2)=X$, and

$$p=P[\phi(X_1, X_2)=1]=P[X=1]=p_1=p_2,$$

where $p_1=P[X_1=1]$ and $p_2=P[X_2=1]$. The system hazard is $u=-\log p$ and the component hazards are $u_1=-\log p_1=u$, $u_2=-\log p_2=u$. If one now tries to approximate $\eta=u$ by $\eta^*=u_1u_2=u^2$, only part (a) of Lemma 2.1 remains valid. In particular if $0<u<1$, then $\eta^*<\eta$, i.e., the approximation is not conservative. □

In a coherent system those combinations of components whose failure is just enough to cause a system failure are called minimal cuts. More precisely, a set of components K is a *cut* if $X_i=0$, $i \in K$ and $X_i=1$, $i \notin K$ implies $\phi(\mathbf{X})=0$, and K is a *minimal cut* if no proper subset of K is also a cut. For example, the system shown in Figure 1 has the four minimal cuts

$$K_1=\{1, 2, 4\}, \quad K_2=\{1, 2, 5\}, \quad K_3=\{3, 4\}, \quad K_4=\{3, 5\}.$$

and the system shown in Figure 2 also has four minimal cuts, but these are

$$K_1=\{1, 2\}, \qquad K_2=\{1, 3, 5\}, \qquad K_3=\{2, 3, 4\}, \qquad K_4=\{4, 5\}.$$

We will denote the minimal cuts of a coherent system by $K_1, \ldots, K_k$, where k is the number of minimal cuts.

Any coherent system can be represented in terms of its minimal cuts by forming, for each minimal cut, a parallel subsystem from the components in the cut, and then connecting the parallel subsystems in series. Formally the system structure function ϕ satisfies

$$\phi(\mathbf{X})=\prod_{j=1}^{k}\phi_j(\mathbf{X}), \tag{2.3}$$

where $\phi_j(\mathbf{X})=1-\prod_{i\in K_j}(1-X_i)$, $j=1, \ldots, k$, are the structure functions of the parallel subsystems corresponding to the minimal cuts of the system. More graphically, the minimal cuts of a coherent system determine a particular way of drawing its block diagram. For example, the block diagram shown in Figure 3 is equivalent to the block diagram shown in Figure 1, and the block diagram of Figure 4 is equivalent to the block diagram of Figure 2.

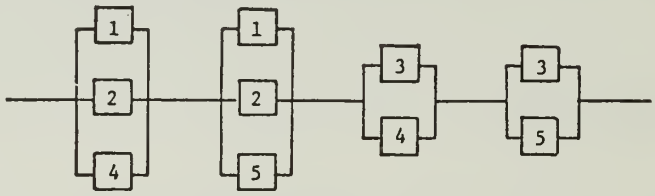

FIGURE 3.

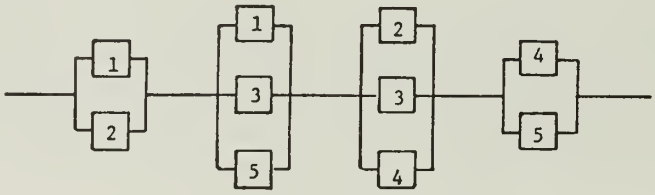

FIGURE 4.

An important thing to note about the minimal cut representation of a coherent system is that the same component can appear in several minimal cuts. So even though the components perform independently, the parallel subsystems corresponding to the minimal cuts in general do not. Thus, if $h_j(\mathbf{p})=1-\prod_{i\in K_j}(1-p_i)$, $j=1, \ldots, k$, are the reliability functions of the parallel subsystems, then the relationship $h(\mathbf{p})=\prod_{j=1}^{k}h_j(\mathbf{p})$, which would hold if the parallel systems performed independently, is *not necessarily valid.* What is true is that

$$h(\mathbf{p})\geq\prod_{j=1}^{k}h_j(\mathbf{p}), \qquad \mathbf{0}\leq\mathbf{p}\leq\mathbf{1}, \tag{2.4}$$

(Esary and Proschan, 1963, Theorem 4.1). The function

$$h_{MC}(\mathbf{p}) = \prod_{j=1}^{k} h_j(\mathbf{p}), \qquad \mathbf{0} \leq \mathbf{p} \leq \mathbf{1}, \tag{2.5}$$

is the *minimal cut lower bound* on the exact reliability function h.

Recall that $\eta(\mathbf{u}) = -\log h(\mathbf{p})$ is the exact system hazard transform. Define the *minimal cut upper bound* on η by

$$\eta_{MC}(\mathbf{u}) = -\log h_{MC}(\mathbf{p}) = \sum_{j=1}^{k} \eta_j(\mathbf{u}), \qquad \mathbf{u} \geq \mathbf{0}, \tag{2.6}$$

where $\eta_j(\mathbf{u}) = -\log h_j(\mathbf{p})$, $j = 1, \ldots, k$. The bound is obtained by finding the exact hazard transform for each of the parallel subsystems in the minimal cut representation for the system and adding these together as if the parallel systems performed independently. The following lemma is largely a re-statement of inequality (2.4).

LEMMA 2.3: *For a coherent system the following comparisons exist between the hazard transform η and η_{MC}, the minimal cut upper bound on η:*

(a) $\eta_{MC}(\mathbf{0}) = \eta(\mathbf{0}) = 0$.

(b) $\eta_{MC}(\mathbf{u}) \geq \eta(\mathbf{u})$, $\mathbf{u} \geq \mathbf{0}$.

PROOF: That $\eta_{MC}(\mathbf{0}) = 0$ follows from (2.6) and part (a) of Lemma 2.1. If $\mathbf{u} = \mathbf{0}$, then $\mathbf{p} = (e^{-0}, \ldots, e^{-0}) = \mathbf{1}$, and $\eta(\mathbf{0}) = -\log h(\mathbf{1}) = -\log 1 = 0$. Thus (a) holds. That (b) holds follows from inequality (2.4), since $\eta_{MC}(\mathbf{u}) = -\log h_{MC}(\mathbf{p}) \geq -\log h(\mathbf{p}) = \eta(\mathbf{u})$. □

Remark 2.4 In contrast with part (b) of Lemma 2.1, part (b) of Lemma 2.3 remains valid, in essence, when component performances are positively dependent in a sense called "association" (Esary, Proschan, and Walkup [2] and Esary-Proschan [3]. However, making use of this fact in the calculation of conservative approximate hazard transforms requires finding suitable modifications, under whatever degree of association is specified, to the approximation (2.2) for parallel systems. □

The *approximate hazard transform* that we consider for an arbitrary coherent system with minimal cuts $K_1, \ldots, K_k$ is

$$\eta^*(\mathbf{u}) = \sum_{j=1}^{k} \eta^*_j(\mathbf{u}), \qquad \mathbf{u} \geq \mathbf{0}, \tag{2.7}$$

where $\eta^*_j(\mathbf{u}) = \prod_{i \in K_j} u_i$, $j = 1, \ldots, k$. The approximation is obtained by finding the approximate hazard transform for each of the parallel subsystems in the minimal cut representation for the system and then adding the results as if the parallel systems performed independently. Thus approximations at two different levels are involved. For example, the approximate hazard transform for the system of Figures 1 and 3 is

$$\eta^*(u_1, \ldots, u_5) = u_1u_2u_4 + u_1u_2u_5 + u_3u_4 + u_3u_5, \tag{2.8}$$

and for the system of Figures 2 and 4

$$\eta^*(u_1, \ldots, u_5) = u_1u_2 + u_1u_3u_5 + u_2u_3u_4 + u_4u_5. \tag{2.9}$$

The following theorem shows that the approximate hazard transform for a coherent system is: (a) exact for perfectly reliable components, and (b) conservative.

THEOREM 2.5: *For a coherent system the following comparisons exist between the hazard transform* η, *the minimal cut upper bound* η_{MC} *on the hazard transform, and the approximate hazard transform* η^*:

(a) $\eta^*(\mathbf{0}) = \eta_{MC}(\mathbf{0}) = \eta(\mathbf{0}) = 0$.

(b) $\eta^*(\mathbf{u}) \geq \eta_{MC}(\mathbf{u}) \geq \eta(\mathbf{u})$, $\mathbf{u} \geq \mathbf{0}$.

PROOF: That $\eta^*(0) = 0$ is immediate from (2.7). That $\eta_{MC}(\mathbf{0}) = \eta(\mathbf{0}) = 0$ is part (a) of Lemma 2.3. Thus (a) holds. To show (b) recall that $\eta_{MC}(\mathbf{u}) \geq \eta(\mathbf{u})$ is part (b) of Lemma 2.3, and note that from (2.6), (2.7), and part (b) of Lemma 2.1

$$\eta^*(\mathbf{u}) = \sum_{j=1}^{k} \eta^*{}_j(\mathbf{u}) \geq \sum_{j=1}^{k} \eta_j(\mathbf{u}) = \eta_{MC}(\mathbf{u}).$$

Thus (b) holds. □

3. A CALCULUS FOR APPROXIMATE HAZARD TRANSFORMS

One of the most attractive features of the approximate hazard transform (2.7) is the simple calculus by which it can be computed for certain important categories of coherent systems, notably the class of systems that can be formed by successive series and parallel combinations of subsystems with non-overlapping sets of components, i.e., the class of *simple* systems considered by Lomnicki [5]. The essential ideas of this calculus are described in section 1, and its application to the system of Figure 1 is illustrated. Note that the approximate hazard transform for that system given in (1.1) agrees with the approximate hazard transform given in (2.8).

For the purposes of this section we will denote a system by a couple (C, ϕ), where C is the set of components used in forming the system (from the mathematical viewpoint C is a set of indices i used to label the components) and ϕ is the structure function of the system. We will be considering coherent systems (C, ϕ) formed from two coherent sybsystems, (C_1, ϕ_1) and (C_2, ϕ_2). In this situation $C = C_1 \cup C_2$, i.e., the system component set consists of all the components appearing in either subsystem. The subsystems have non-overlapping or *disjoint* component sets if $C_1 \cap C_2 = \emptyset$, where $\emptyset$ is the empty set, i.e., no component appears in both subsystems. The reason for being interested in subsystems with disjoint component sets is that *if all the components in* C *perform independently and* C_1, C_2 *are disjoint, then the subsystems perform independently.*

The system (C, ϕ) is a *series* combination of the subsystems (C_1, ϕ_1) and (C_2, ϕ_2) if

$$\phi(\mathbf{X}) = \phi_1(\mathbf{X})\phi_2(\mathbf{X}), \qquad \text{for all realizations of } \mathbf{X}, \tag{3.1}$$

where $\mathbf{X}$ is the vector of component performance indicators X_i, $i \epsilon C$. If C_1 and C_2 are disjoint, then ϕ_1 depends only on those X_1 such that $i \epsilon C_1$ and ϕ_2 depends only on those X_i such that $i \epsilon C_2$. Similarly, (C, ϕ) is a *parallel* combination of (C_1, ϕ_1) and (C_2, ϕ_2) if

(3.2) $$\phi(\mathbf{X}) = 1 - \{1 - \phi_1(\mathbf{X})\}\{1 - \phi_2(\mathbf{X})\}, \quad \text{for all realizations of } \mathbf{X}.$$

It is easy to check that a series or parallel combination of coherent subsystems is a coherent system. Graphically, these definitions correspond to being able to display the block diagram for the system as a series or parallel combination of the block diagrams for the subsystems. The following proposition describes how approximate hazard transforms can be computed, for series and parallel combinations of subsystems with disjoint component sets, to obtain results that agree with (2.7).

PROPOSITION 3.1: *Suppose a coherent system* (C, ϕ) *is a combination of the coherent subsystems* (C_1, ϕ_1) *and* (C_2, ϕ_2) *where* C_1 and C_2 *are disjoint.* Let η^* *be the appropriate hazard transform for* (C, ϕ) *and* η_1^*, η_2^* *be the approximate hazard transforms for* (C_1, ϕ_1) *and* (C_2, ϕ_2). *Then for all* $\mathbf{u} \geq \mathbf{0}$:

(a) $\eta^*(\mathbf{u}) = \eta_1^*(\mathbf{u}) + \eta_2^*(\mathbf{u})$, *if the combination is series.*

(b) $\eta^*(\mathbf{u}) = \eta_1^*(\mathbf{u})\, \eta_2^*(\mathbf{u})$, *if the combination is parallel.*

PROOF: Suppose (C_1, ϕ_1) has the minimal cuts $K_{11}, \ldots, K_{1k_1}$ and (C_2, ϕ_2) has the minimal cuts $K_{21}, \ldots, K_{2k_2}$.

(a) If the combination is series, then since C_1 and C_2 are disjoint, (C, ϕ) has $k = k_1 + k_2$ minimal cuts, namely

$$K_1 = K_{11}, \ldots, K_{k_1} = K_{1k_1}, K_{k_1+1} = K_{21}, \ldots, K_{k_1+k_2} = K_{2k_2}.$$

From (2.7)

$$\eta^*(\mathbf{u}) = \sum_{j=1}^{k} \prod_{i \in K_j} u_i = \sum_{j_1=1}^{k_1} \prod_{i \in K_{1j_1}} u_i + \sum_{j_2=1}^{k_2} \prod_{i \in K_{2j_2}} u_i$$

$$= \eta_1^*(\mathbf{u}) + \eta_2^*(\mathbf{u}),$$

so (a) holds.

(b) If the combination is parallel, then since C_1 and C_2 are disjoint, (C, ϕ) has $k = k_1 k_2$ minimal cuts, namely all $K_j = K_{1j_1} \cup K_{2j_2}$, where $j = 1, \ldots, k$ as $j_1 = 1, \ldots, k_1$, and $j_2 = 1, \ldots, k_2$. From (2.7)

$$\eta^*(\mathbf{u}) = \sum_{j=1}^{k} \prod_{i \in K_j} u_i = \sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \prod_{i \in K_{1j_1} \cup K_{2j_2}} u_i$$

$$= \sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \prod_{i \in K_{1j_1}} u_i \cdot \prod_{i \in K_{2j_2}} u_i$$

$$= \sum_{j_1=1}^{k_1} \prod_{i \in K_{1j_1}} u_i \cdot \sum_{j_2=1}^{k_2} \prod_{i \in K_{2j_2}} u_i = \eta_1^*(\mathbf{u}) \eta_2^*(\mathbf{u}),$$

so (b) holds. □

It is clear that if a system is formed by a sequence of series or parallel combinations of *modules*, i.e., subsystems with non-overlapping component sets, then parts (a) and (b) of Proposition 3.1 can be applied in the same sequence to evaluate its approximate hazard transform. Thus the proposition defines a calculus which is applicable to simple systems.

REMARK 3.2: Proposition 3.2 can be extended to show that whenever a module (cf. Birnbaum and Esary [1]) occurs in a coherent system, the approximate transform for the system can be found by first finding the approximate hazard transform for the module and then proceeding as if the module were a component in the larger system with the hazard given by its approximate transform. □

A specialized application of the approximate hazard transform is found in the *Guide Manual for Reliability Measurement* [8, section 3.1.6]. All components are assumed to have exponential life distributions, i.e., for a mission of duration t, $p_i = e^{-\lambda_i t}$, $i = 1, \ldots, n$. The mission length is taken as the unit of time measurement, and the component failure rates λ_i are scaled accordingly. In this way the mission reliability of a component has the simple form $p_i = e^{-\lambda_i}$. The component mission hazard is then $u_i = \lambda_i$. Under these conventions the calculus used in the manual is a special case of the approximate hazard transform calculus.

4. THE FIRST-ORDER APPROXIMATION

The approximate hazard transform η^* is always conservative. In this section we describe a *first-order approximate hazard transform* η^1 which is easier to compute than η^*, sometimes more accurate than η^*, but is not always conservative.

The approximate hazard transform η^* is based on all minimal cuts of the system, i.e., η^*, computed from (2.7), is the sum of the products of component hazards over each minimal cut. The first-order approximation is computed in a similar manner but using only the minimal cuts having the smallest number of components. The procedure is best illustrated by an example.

Recall that the system shown in Figures 2 and 4 has four minimal cuts. Two minimal cuts, $K_1 = \{1, 2\}$ and $K_4 = \{4, 5\}$, contain two components. The other cuts contain more than two components. The first-order approximation for this system is based only on cuts K_1 and K_2, i.e.

$$\eta^1(u_1, \ldots, u_5) = u_1u_2 + u_4u_5. \tag{4.1}$$

The procedure for the first-order approximation, then, is to identify the smallest minimal cuts and to sum the products of component hazards over those cuts. By ignoring the larger minimal cuts, the first-order approximation tends to "correct" for the conservative error in the approximate hazard transform. Of course the "correction" may be too large, i.e., the first-order approximation is not always conservative.

If a component does not appear in any of the smallest minimal cuts, e.g., component 3 of Figures 2 and 4, then the first-order approximation implicitly treats the component as being perfectly reliable. If the reliability of such components is no less than the reliability of the components in the smallest minimal cuts, then the first-order approximation usually gives rather good results.

5. NUMERICAL COMPARISONS

In this section we compute system hazard using the approximate hazard transform η^* and the first-order approximation η^1, and compare the results with the true system hazard η for selected

examples. The comparison is shown in terms of "percent error" where

$$\% \text{ error} = \frac{(\eta^1 \text{ or } \eta^*) - \eta}{\eta} \times 100.$$

In the examples, and in the experience of the authors with similar examples involving small systems, the accuracy of the approximations usually decreases with decreasing component reliabilities. If the reliabilities are greater than 0.9, the approximations usually are in error by less than 20 percent.

The formulas for the true system hazard are not exhibited in the examples. Some appreciation of the need for approximations can be gained by working them out. The motivation for the approximations discussed here is not only that numerical calculations are less tedious, but also that a formula for numerical calculations, or other purposes, can be derived with relative ease.

EXAMPLE 5.1: Suppose the components in the system of Figures 1 and 3 have independent times to failure, exponentially distributed with parameters $\lambda_1, \ldots, \lambda_5$. Then for a mission of duration t, $p_i = e^{-\lambda_i t}$ and $u_i = \lambda_i t$, $i = 1, \ldots, 5$. The approximations to the system hazard for the mission are

$$\eta^1 = u_3u_4 + u_3u_5 = \lambda_3 t\lambda_4 t + \lambda_3\lambda_5 t = \lambda_3(\lambda_4 + \lambda_5)t^2,$$

and

$$\eta^* = (\lambda_1\lambda_2 t^2 + \lambda_3 t)(\lambda_4 t + \lambda_5 t)$$

[cf. (1.1)]. Some numerical comparisons for this system are tabulated below:

CASE 1: $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0.10$.

t	η	Percent error	
		η^1	η^*
0.2	0.0008	1.2	2.4
0.4	0.0031	2.2	6.0
0.6	0.0070	3.3	9.5
0.8	0.0122	4.5	12.8
1.0	0.0189	5.8	16.4

CASE 2: $\lambda_1 = \lambda_2 = \lambda_3 = 0.20$, $\lambda_4 = \lambda_5 = 0.10$.

t	η	Percent error	
		η^1	η^*
0.2	0.0016	—	4.4
0.4	0.0064	0.8	8.8
0.6	0.0142	1.6	13.8
0.8	0.0249	2.7	19.1
1.0	0.0385	4.0	24.7

In Case 2, when $t=1.0$, $p_i=0.82$, $i=1, 2, 3$. This low component reliability adversely affects the accuracy of the approximate hazard transform. □

EXAMPLE 5.2: Let components 1, 2, 3, 4 in the system of Figures 2 and 4 have independent times to failure with the distributions shown below. Let component 3 be a "one-shot" device which, independently, either functions for a mission of any duration or is failed from the outset.

Component	Distribution	Parameters	Component hazards for a mission of duration t
1	Exponential	λ	λt
2	Exponential	λ	λt
3	Bernoulli	p_3	$-\log p_3$
4	Weibull	μ, α	νt^{α}
5	Weibull	μ, α	νt^{α}

For a mission of duration t

$$\eta^1=\lambda^2t^2+\mu^2t^{2\alpha}$$

[cf. (4.1)], and

$$\eta^*=\lambda^2t^2-(\log p_3)\ 2\lambda\mu t^{\alpha+1}+\mu^2t^{2\alpha}$$

[cf. (2.9)]. Some numerical comparisons are tabulated below:

CASE 1: $\lambda=0.10$, $p_3=0.90$, $\mu=0.10$, $\alpha=2$.

t	η	Percent error	
		η^1	η^*
0.2	0.0004	−1.5	2.4
0.4	0.0019	−2.7	4.3
0.6	0.0050	−2.5	6.5
0.8	0.0106	−1.1	9.1
1.0	0.0197	+1.5	12.2

CASE 2: $\lambda=0.10$, $p_3=0.95$, $\mu=0.15$, $\alpha=2$.

t	η	Percent error	
		η^1	η^*
0.2	0.0006	0.1	2.5
0.4	0.0030	0.7	4.8
0.6	0.0081	2.3	7.4
0.8	0.0179	5.2	10.8
1.0	0.0342	9.5	15.2

In Case 1 the first-order approximation is not conservative for $t \leq 0.8$, i.e., system reliability is *over*-estimated. Note that the component ignored by the approximation, component 3, is the least reliable component in the system for $t \leq 1.0$. □

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TESTING FIT WITH NUISANCE LOCATION AND SCALE PARAMETERS

Lionel Weiss*

Cornell University

ABSTRACT

For each n, $X_1(n), \ldots, X_n(n)$ are independent and identically distributed random variables, each with cumulative distribution function $F(x)$ which is known to be absolutely continuous but is otherwise unknown. The problem is to test the hypothesis that $F(x) = G\left(\frac{x-\theta_1}{\theta_2}\right)$, where the cumulative distribution function $G(x)$ is completely specified and satisfies certain regularity conditions, and the parameters θ_1, θ_2 are unknown and unspecified, except that the scale parameter θ_2, is positive. $Y_1(n) \leqq Y_2(n) \leqq \ldots \leqq Y_n(n)$ are the ordered values of $X_1(n), \ldots, X_n(n)$. A test based on a certain subset of $\{Y_i(n)\}$ is proposed, is shown to have asymptotically a normal distribution when the hypothesis is true, and is shown to be consistent against all alternatives satisfying a mild regularity condition.

1. INTRODUCTION AND NOTATION

For each positive integer n, $X_1(n), \ldots, X_n(n)$ are independent and identically distributed random variables, with common cumulative distribution function $F(x)$. $F(x)$ is known to be absolutely continuous with density function denoted by $f(x)$, but is otherwise unknown. $Y_1(n) \leqq Y_2(n) \leqq \ldots \leqq Y_n(n)$ are the ordered values of $X_1(n), \ldots, X_n(n)$. From now on, for typographical simplicity we write X_i instead of $X_i(n)$, and Y_i instead of $Y_i(n)$.

The problem is to test the hypothesis that $F(x) = G\left(\frac{x-\theta_1}{\theta_2}\right)$ for all x, where $G(x)$ is a completely specified cumulative distribution function with density $g(x)$, and θ_1, θ_2 are unknown and unspecified parameters with $\theta_2 > 0$. $G(x)$ satisfies certain regularity conditions to be specified below.

Define $\alpha_n = \frac{1}{2}(\log n)^{1/4}$, $\Delta_n = 1 - (\log n)^{-1/4}$, $L_n = n^{\Delta_n}$, $\bar{K}_n = n^{1-\Delta_n}\{1 - 2(n^{\Delta_n - 1})^{1/2\alpha_n}\}$. If n is even, define L_n as the largest integer no greater than $\bar{L}_n$, and K_n as the largest even integer no greater than $\bar{K}_n$. If n is odd, define L_n as the largest odd integer no greater than $\bar{L}_n$, and K_n as the largest odd integer no greater than $\bar{K}_n$. Thus $n - K_n L_n$ is always an even integer. Define p_n by the equation $K_n L_n = n - 2np_n$, so np_n is an integer. Define q_n as $1 - p_n$. Then nq_n is an integer. From the definitions, we can easily show the following:

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$$\lim_{n\to\infty}\frac{L_n}{n^{\Delta_n}}=1$$

$$\lim_{n\to\infty}\frac{K_n}{n^{1-\Delta_n}}=1 \tag{1.1}$$

$$\lim_{n\to\infty}\{p_n n^{(\log n)^{-1/2}}\}=1.$$

Define B_1' as $glb\{x\colon G(x)>0\}$, and B_2' as $lub\{x\colon G(x)<1\}$. B_1' may be $-\infty$, B_2' may be ∞. Define β_n as $\inf_x \{g(x)\colon G^{-1}(p_n)\leqq x\leqq G^{-1}(q_n)\}$. We assume the following:

(1.2) For all x in (B_1', B_2'), $g'(x)$ exists and $|g'(x)|<D<\infty$.

(1.3) $$g(x)<B<\infty \quad \text{for all } x.$$

(1.4) $$|x|g(x)<C<\infty \quad \text{for all } x.$$

(1.5) There exist values ρ_1, ρ_2, with $B_1'\leqq\rho_1<\rho_2\leqq B_2'$, such that $g(x)>\bar{\rho}>0$ for all x in (ρ_1,ρ_2), $g(x)$ is nondecreasing in (B_1',ρ_1), and $g(x)$ is nonincreasing in (ρ_2, B_2').

(1.6) $$\lim_{n\to\infty}\frac{n\beta_n^2}{L_n}=\infty.$$

(1.7) $$\int_{-\infty}^{\infty}\left[y\frac{g'(y)}{g(y)}\right]^2 g(y)\,dy<\infty.$$

These conditions are satisfied by the normal, uniform, and exponential distributions, as well as most other common continuous distributions.

For $j=1,\ldots,K_n+1$, denote $G^{-1}\left(\frac{np_n+(j-1)L_n}{n}\right)$ by $U(j,n)$, and $g(U(j,n))$ by $u(j,n)$. We note that if $F(x)=G\left(\frac{x-\theta_1}{\theta_2}\right)$ for all x, then $F^{-1}(t)=\theta_1+\theta_2 G^{-1}(t)$, and $f(F^{-1}(t))=\frac{1}{\theta_2}g(G^{-1}(t))$ for all t in (0,1). For $j=1,\ldots,K_n+1$, denote

$$\frac{\sqrt{n}\,u(j,n)}{\theta_2}\left(Y_{np_n+(j-1)L_n}-\theta_1-\theta_2 U(j,n)\right)$$

by $Z_j(n;\theta_1,\theta_2)$. For any region $R(K_n+1)$ in the space of $\{Z_1(n;\theta_1,\theta_2),\ldots,Z_{K_n+1}(n;\theta_1,\theta_2)\}\equiv \vec{Z}(n;\theta_1,\theta_2)$, let $\bar{G}_n(R(K_n+1);\theta_1,\theta_2)$ denote the probability that $\vec{Z}(n;\theta_1,\theta_2)$ falls in $R(K_n+1)$ when $G\left(\frac{x-\theta_1}{\theta_2}\right)$ is the true cumulative distribution function for X_i. Let $h_n(z_1,\ldots,z_{K_n+1})$ denote the following (K_n+1) — variate normal density function:

$$\frac{C_n}{(2\pi)^{(1/2)(K_n+1)}}\exp\left\{-\frac{n(L_n-1)}{2L_n^2}\left(\frac{L_n}{np_n}(z_1^2+z_{K_n+1}^2)+\sum_{j=1}^{K_n}(z_{j+1}-z_j)^2\right)\right\},$$

where C_n is the appropriate constant and is given explicitly in [5]. The means in this distribution are

all zero, and the variances and covariances are given explicitly in [5]. Let $\bar{H}_n(R(K_n+1))$ denote the probability assigned to the region $R(K_n+1)$ by the density $h_n(z_1, \ldots, z_{k_n+1})$.

LEMMA 1.1: For each n, suppose $R(K_n+1)$ is an arbitrary measurable region in (K_n+1)-dimensional space. Then $\lim_{n\to\infty} | \bar{G}_n(R(K_n+1);\ \theta_1, \theta_2) - \bar{H}_n(R(K_n+1)) | = 0$.

PROOF: This lemma is the Theorem of [5], so the proof is merely the verification that conditions (1.1) to (1.19) of [5] hold in our present case. Before discussing the verification, we note the following changes in notation between [5] and the present paper:

Present symbol	*Corresponding symbol in [5]*	*Present symbol*	*Corresponding symbol in [5]*
$G\left(\frac{x-\theta_1}{\theta_2}\right)$	$F_n(x)$	$\frac{1}{\theta_2}B$	B_n
$\frac{1}{\theta_2}g\left(\frac{x-\theta_1}{\theta_2}\right)$	$f_n(x)$	$\frac{1}{\theta_2^2}D$	D_n
$\theta_1+\theta_2 B_1'$	$\bar{B}_1(n)$	$\theta_1+\theta_2 U(j, n)$	$T(j, n)$
$\theta_1+\theta_2 B_2'$	$\bar{B}_2(n)$	$\frac{1}{\theta_2}u(j, n)$	$t(j, n)$
$\frac{1}{\theta_2}\beta_n$	b_n		

In the proof of the lemma, we distinguish two cases:

CASE 1: $\lim_{n\to\infty} \beta_n = 0$. Then for all sufficiently large n, $\frac{1}{\theta_2}\beta_n \leqq 1$, so assumption (1.3) of [5] is satisfied. Verifying that the other assumptions of [5] are satisfied is simple, with the possible exception of (1.18) and (1.19) of [5]. (1.18) of [5] becomes, in our present notation

$$\lim_{n\to\infty}\left\{ \sqrt{\frac{n}{p_n}}\, g(G^{-1}(p_n))\,(B_1'-G^{-1}(p_n)) \right\} = 0. \tag{1.8}$$

Since $B_1' = G^{-1}(0)$, we can write $B_1' - G^{-1}(p_n) = \frac{-p_n}{g(G^{-1}(\omega p_n))}$, where $0<\omega<1$. If $\rho_1 = B_1'$, the expression in braces in (1.8), for all sufficiently large n, is not greater than $-\sqrt{np_n}\,\frac{\bar{\rho}}{B}$, which approaches $-\infty$ as n increases. If $\rho_1 > B_1'$, then for all sufficiently large n the expression in braces in (1.8) is not greater than $-\sqrt{np_n}$, which approaches $-\infty$ as n increases. Thus (1.18) of [5] is true. The demonstration of (1.19) of [5] is similar.

CASE 2: It is not true that $\lim_{n\to\infty} \beta_n = 0$. Then assumption (1.3) of [5] is not necessarily satisfied. However, the Theorem of [5] is still true. Assumption (1.3) of [5] was merely a convenience to avoid having to modify certain statements in the proof of the Theorem of [5]. Thus Lemma 1.1 is true in Case 2 also.

2. CONSTRUCTION OF THE TEST

With no loss of generality, we assume $G^{-1}\left(\frac{1}{2}\right)=0$, $G^{-1}\left(\frac{3}{4}\right)-G^{-1}\left(\frac{1}{4}\right)=1$. This can be achieved by a linear transformation of parameters if necessary, and from now on we assume that it is true.

For any given values w_1, w_2, with $w_2 > 0$, define $W_n(w_1, w_2)$ as

$$\frac{n(L_n-1)}{L_n^2}\left(\frac{n}{w_2^2}\right)\left[\begin{array}{l}\dfrac{L_n}{np_n}\left\{\begin{array}{l}u^2(1,n)\,(Y_{np_n}-w_1-w_2U(1,n))^2\\+u^2(K_n+1,n)\,(Y_{nq_n}-w_1-w_2U(K_n+1,n))^2\end{array}\right\}\\[2ex]\quad+\displaystyle\sum_{j=1}^{K_n}\left\{\begin{array}{l}u(j+1,n)\,(Y_{np_n+jL_n}-w_1-w_2U(j+1,n))\\-u(j,n))Y_{np_n+(j-1)L_n}-w_1-w_2U(j,n))\end{array}\right\}^2\end{array}\right]$$

We note that from the definition of $Z_j(n;\theta_1,\theta_2)$ and Lemma 1.1, it follows that when the actual distribution function of X_1 is $G\left(\frac{x-\theta_1}{\theta_2}\right)$, then for all asymptotic probability calculations we can assume that $W_n(\theta_1,\theta_2)$ has a chi-square distribution with (K_n+1) degrees of freedom.

For each n, let $\hat{\theta}_1(n)$ denote the sample median of the sample $(X_1, \ldots, X_n)$, and let $\hat{\theta}_2(n)$ denote the sample interquartile range (the difference between the third quartile and the first quartile) of the sample $(X_1, \ldots, X_n)$. We note that $W_n(\hat{\theta}_1(n), \hat{\theta}_2(n))$ can be computed from the observations, and does not depend on any unknown parameters. Our proposed test rejects the hypothesis when $W_n(\hat{\theta}_1(n), \hat{\theta}_2(n))$ is "too large."

For the remainder of this section, we assume that the hypothesis is true (with θ_1, θ_2 unknown), and we examine the distribution of $W_n(\hat{\theta}_1(n), \hat{\theta}_2(n))$.

From now on, we write Z_j instead of $Z_j(n;\theta_1,\theta_2)$.

Define $R_1(n)$ as $\sqrt{n}(\hat{\theta}_1(n)-\theta_1)$, and $R_2(n)$ as $\sqrt{n}(\hat{\theta}_2(n)-\theta_2)$. Asymptotically, $R_1(n)$, $R_2(n)$ have a joint normal distribution with zero means and finite variances.

Simple calculations show that we can write $W_n(\hat{\theta}_1(n),\hat{\theta}_2(n))$ as

$$\frac{n(L_n-1)}{L_n^2(\hat{\theta}_2(n))^2}\left\{\frac{L_n^2\theta_2^2}{n(L_n-1)}W_n(\theta_1,\theta_2)+Q(n)\right\}, \tag{2.1}$$

where $Q(n)$ is the sum of the following four expressions:

$$\frac{-2L_n\theta_2}{np_n}\left\{\begin{array}{l}[R_1(n)+R_2(n)U(1,n)]u(1,n)Z_1\\+[R_1(n)+R_2(n)U(K_n+1,n)]u(K_n+1,n)Z_{K_n+1}\end{array}\right\} \tag{2.2}$$

$$\frac{L_n}{np_n}\left\{\begin{array}{l}u^2(1,n)[R_1(n)+R_2(n)U(1,n)]^2\\+u^2(K_n+1,n)[R_1(n)+R_2(n)U(K_n+1,n)]^2\end{array}\right\} \tag{2.3}$$

$$(2.4)\qquad \sum_{j=1}^{K_n} \{R_1(n)[u(j+1,n)-u(j,n)]+R_2(n)[u(j+1,n)U(j+1,n)-u(j,n)U(j,n)]\}^2$$

$$(2.5)\qquad -2\theta_2 \sum_{j=1}^{K_n} \left\{ (Z_{j+1}-Z_j) \begin{pmatrix} R_1(n)[u(j+1,n)-u(j,n)]+ \\ R_2(n)[u(j+1,n)U(j+1,n)-u(j,n)U(j,n)] \end{pmatrix} \right\}.$$

Define $\gamma(n)$ as $\left[\frac{L_n^2\sqrt{K_n+1}}{n(L_n-1)}\right]^{-1} Q(n)$. We now show that $\gamma(n)$ converges stochastically to zero as n increases.

Using Lemma 1.1, we can consider Z_1 and Z_{K_n+1} as normal, with zero means, and variances, given explicitly in [5], which approach zero as n increases. Then, using (1.1), (1.3), and (1.4), we see that with probability approaching one as n increases, the absolute value of expression (2.2) is less than $\frac{L_n}{np_n}$, and $\frac{L_n}{np_n}\left[\frac{L_n^2\sqrt{K_n+1}}{n(L_n-1)}\right]^{-1}$ approaches zero as n increases.

It is easily verified that | expression (2.3) | $\left[\frac{L^2\sqrt{K_n+1}}{n(L_n-1)}\right]^{-1}$ converges stochastically to zero as n increases.

Define $S_1(n)$ as $\sum_{j=1}^{K_n} [u(j+1, n)-u(j, n)]^2$,

$$S_2(n) \text{ as } \sum_{j=1}^{K_n} \{[u(j+1, n)U(j+1, n)-u(j, n)U(j, n)]\}^2, \text{ and}$$

$$S_3(n) \text{ as } \sum_{j=1}^{K_n} \{[u(j+1, n)-u(j, n)]\,[u(j+1, n)U(j+1, n)-u(j, n)U(j, n)]\}.$$

Then (2.4) can be written as $R_1^2(n)S_1(n)+R_2^2(n)S_2(n)+2R_1(n)R_2(n)S_3(n)$. It is easily verified that $S_1(n)=\frac{L_n}{n}\left\{\int_{-\infty}^{\infty}\left(\frac{g'(x)}{g(x)}\right)^2 g(x)dx+\epsilon_1(n)\right\}$,

$$S_2(n)=\frac{L_n}{n}\left\{\int_{-\infty}^{\infty}\left\{1+y\frac{g'(y)}{g(y)}\right\}^2 g(y)dy+\epsilon_2(n)\right\}, \text{ and}$$

$S_3(n)=\frac{L_n}{n}\left\{\int_{-\infty}^{\infty}\left(\frac{g'(y)}{g(y)}\right)\left\{1+y\frac{g'(y)}{g(y)}\right\} g(y)dy+\epsilon_3(n)\right\}$, where $\epsilon_1(n)$, $\epsilon_2(n)$, and $\epsilon_3(n)$ all approach zero as n increases. By (1.7), all three integrals are finite. It follows directly that | expression (2.4) | $\left[\frac{L_n^2\sqrt{K_n+1}}{n(L_n-1)}\right]^{-1}$ converges stochastically to zero as n increases.

Define $S_4(n)$ as $\sum_{j=1}^{K_n} [u(j+1, n)-u(j, n)](Z_{j+1}-Z_j)$, and $S_5(n)$ as $\sum_{j=1}^{K_n} [u(j+1, n)U(j+1, n)-u(j, n)U(j, n)](Z_{j+1}-Z_j)$. Then (2.5) can be written as $-2\theta_2[R_1(n)S_4(n)+R_2(n)S_5(n)]$. By Lemma 1.1, for all asymptotic probability calculations we can assume $Z_1, \ldots, Z_{K_n+1}$ have joint

density $h_n(z_1, \ldots, z_{K_n+1})$. Under this assumption, using the formula on page 99 of [5], we find that for $i<j$, covariance $(Z_{j+1}-Z_j, Z_{i+1}-Z_i)=-\frac{L_n}{nK_n}(1+\delta'_n)$, where δ'_n does not depend on i or j and $\lim_{n\to\infty}\delta'_n=0$, and variance $(Z_{j+1}-Z_j)=\frac{L_n}{n}(1+\delta''_n)$, where δ''_n does not depend on j and $\lim \delta''_n=0$. It follows that for all asymptotic probability calculations we can assume that $S_4(n)$ has a normal distribution with mean zero and variance

$$\frac{L_n}{n}(1+\delta''_n)\sum_{j=1}^{K_n}[u(j+1,n)-u(j,n)]^2$$

(2.6)

$$-\frac{2L_n}{nK_n}(1+\delta'_n)\sum_{j=2}^{K_n}\sum_{i=1}^{j-1}[u(j+1,n)-u(j,n)][u(i+1,n)-u(i,n)],$$

and we can assume that $S_5(n)$ has a normal distribution with mean zero and variance

$$\frac{L_n}{n}(1+\delta''_n)\sum_{j=1}^{K_n}[u(j+1,n)U(j+1,n)-u(j,n)U(j,n)]^2$$

(2.7)

$$-\frac{2L_n}{nK_n}(1+\delta'_n)\sum_{j=2}^{K_n}\sum_{i=1}^{j-1}[u(j+1,n)U(j+1,n)-u(j,n)U(j,n)]$$
$$[u(i+1,n)U(i+1,n)(i,n)U(i,n)].$$

It is easily seen that $\frac{\{\text{expression (2.6)}\}}{(L_n/n)^2}$ and $\frac{\{\text{expression (2.7)}\}}{(L_n/n)^2}$ both remain bounded as n increases,

Then with probability approaching one as n increases, $\frac{|S_i(n)|}{(L_n/n)}$ remains bounded, for $i=4, 5$. This means that $\left[\frac{L_n^2\sqrt{K_n+1}}{n(L_n-1)}\right]^{-1}$ |expression (2.5)| converges stochastically to zero as n increases.

Using the results about (2.2)–(2.5) developed above, we see that $\gamma(n)$ converges stochastically to zero as n increases.

Define T_n as $\frac{W_n(\theta_1,\theta_2)-(K_n+1)}{\sqrt{2(K_n+1)}}$. When the hypothesis is true with θ_1, θ_2 as the parameters, T_n has a standard normal distribution asymptotically.

Using (2.1), we can write

$$\frac{W_n(\hat\theta_1(n),\hat\theta_2(n))-(K_n+1)}{\sqrt{2(K_n+1)}}=\frac{1}{\sqrt2}\left\{\left[1+\frac{R_2(n)}{\theta_2\sqrt n}\right]^{-2}\sqrt{K_n+1}-\sqrt{K_n+1}\right\} \tag{2.8}$$

$$+\left[1+\frac{R_2(n)}{\theta_2\sqrt n}\right]^{-2}T_n+\frac{\gamma(n)}{\sqrt2}$$

The right side of (2.8) can be written as $T_n+\bar{\gamma}(n)$, where $\bar{\gamma}(n)$ converges stochastically to zero as n increases.

We have shown that when the hypothesis is true, the asymptotic distribution of

$$\frac{W_n(\hat{\theta}_1(n), \hat{\theta}_2(n)) - (K_n+1)}{\sqrt{2(K_n+1)}}$$

is standard normal. For a value α in (0, 1), define $t(\alpha)$ as satisfying $\int_{t(\alpha)}^{\infty} (2\pi)^{-1/2} e^{-x^2/2} dx = \alpha$. If we want level of significance α, we achieve it approximately by rejecting if $W_n(\hat{\theta}_1(n), \hat{\theta}_2(n)) > (K_n+1) + \sqrt{2(K_n+1)}\, t(\alpha)$.

3. THE CONSISTENCY OF THE TEST

Throughout this section, we assume that the common cumulative distribution function for X_i is $\bar{F}(x)$, with corresponding density $\bar{f}(x)$, and there do not exist values θ_1, θ_2 such that $\bar{F}(x) = G\left(\frac{x-\theta_1}{\theta_2}\right)$. That is, we assume the hypothesis is not true. We also assume that if $\bar{f}(x)$ has any discontinuities, they are all ordinary discontinuities. This is a mild regularity condition.

$\bar{F}^{-1}(0)$ denotes the largest value of x such that $\bar{F}(x) = 0$, and $\bar{F}^{-1}(1)$ denotes the smallest value of x such that $\bar{F}(x) = 1$. For $j = 1, \ldots, K_n+1$, define $\bar{W}_j(n)$ as $\sqrt{n}\left\{\bar{F}(Y_{np_n+(j-1)L_n}) - \frac{np_n+(j-1)L_n}{n}\right\}$. It is well known that $\max_{1 \leqq j \leqq K_n+1} |\bar{W}_j(n)|$ is bounded with probability one.

First, suppose there is a value $\bar{x}$ in the open interval $(\bar{F}^{-1}(0), \bar{F}^{-1}(1))$ such that $\bar{f}(\bar{x}) = 0$. Since we are assuming $\bar{f}(x)$ has only ordinary discontinuities, there is no loss of generality in assuming that $\bar{f}(x)$ is continuous on the left at $\bar{x}$. Then either: (a) There exists a value $\Delta > 0$ such that $\bar{f}(x) = 0$ for all x in $[\bar{x}-\Delta, \bar{x}]$+ or (*b*) There exists a $\Delta > 0$ such that $\bar{f}(x) > 0$ for $\bar{x}-\Delta \leqq x < \bar{x}$, and $\bar{F}(\bar{x}-\Delta) > 0$.

In case (a), since $0 < \bar{F}(\bar{x}) < 1$, there exist finite positive values Γ_1, Γ_2, such that

$$\lim_{n\to\infty} P\left[\begin{array}{l}\text{For some } j \text{ between 2 and } K_n,\ Y_{np_n+jL_n} - Y_{np_n+(j-1)L_n} \geqq \Delta, \\ |\ Y_{np_n+jL_n}| < \Gamma_1, |Y_{np_n+(j-1)L_n}| < \Gamma_1,\ u(j+1, n) > \Gamma_2\end{array}\right] = 1.$$

For such a j, it is easily verified that

$$D(j, n) \text{ defined as } \frac{n}{L_n}\left|\begin{array}{l}u(j+1, n)\ (Y_{np_n+jL_n} - \hat{\theta}_1(n) - \hat{\theta}_2(n)U(j+1, n)) \\ -u(j, n)\ (Y_{np_n} + (j-1)L_n - \hat{\theta}_1(n) - \hat{\theta}_2(n)U(j, n))\end{array}\right|$$

approaches infinity as n increases. Since $W_n(\hat{\theta}_1(n), \hat{\theta}_2(n)) > \frac{n^2(L_n-1)}{L_n^2(\hat{\theta}_2(n))^2}\left(\frac{L_n}{n}\right)^2 D^2(j, n)$, and $\hat{\theta}_2(n)$ remains bounded with probability one as n increases, it follows that $\frac{W_n(\hat{\theta}_1(n), \hat{\theta}_2(n))}{K_n+1}$ approaches infinity stochastically as n increases. Considering the critical value of the test, this proves consistency in case (a).

We investigate case (b). We have $\bar{F}(Y_{np_n+jL_n}) - \bar{F}(Y_{np_n+(j-1)L_n}) = \frac{L_n}{n} + \frac{\bar{W}_{j+1}(n) - \bar{W}_j(n)}{\sqrt{n}}$, and $\bar{F}(Y_{np_n+jL_n}) - \bar{F}(Y_{np_n+(j-1)L_n}) = \bar{f}(\bar{y}(j,n))\ (Y_{np_n+jL_n} - Y_{np_n+(j-1)L_n})$, where $Y_{np_n+(j-1)L_n} < \bar{y}(j,n) < Y_{np_n+jL_n}$. With probability approaching one as n increases, some $\bar{y}(j,n)$ will be below $\bar{x}$, but within δ of it, for any given positive δ. It follows that there exist finite positive values Γ_3, Γ_4 such that for any $\epsilon > 0$,

$$\lim_{n\to\infty} P \left[\begin{array}{l} \text{For some } j \text{ between 2 and } K_n,\ Y_{np_n+jL_n} - Y_{np_n+(j-1)L_n} \geqq \dfrac{L_n}{n\epsilon}, \\ \\ |Y_{np_n+jL_n}| < \Gamma_3,\ |Y_{np_n+(j-1)L_n}| < \Gamma_3,\ u(j+1,n) > \Gamma_4 \end{array} \right] = 1.$$

Defining $D(j,n)$ as in case (a), it follows that $\max_j D(j,n)$ converges stochastically to ∞ as n increases. The proof of consistency follows as in case (a).

The only remaining possibility is that $\bar{f}(x)$ is positive for all x in $(\bar{F}^{-1}(0), \bar{F}^{-1}(1))$. From now on we assume this. Let $\bar{\theta}_1, \bar{\theta}_2$ denote the population median and population interquartile range for $\bar{F}(x)$. $\bar{\theta}_1, \bar{\theta}_2$ are uniquely defined. Define $\bar{R}_1(n) = \sqrt{n}\,(\hat{\theta}_1(n) - \bar{\theta}_1)$, $\bar{R}_2(n) = \sqrt{n}\,(\hat{\theta}_2(n) - \bar{\theta}_2)$. $\bar{R}_1(n)$, $\bar{R}_2(n)$ have asymptotically a joint normal distribution with zero means and finite variances. Let p, q be values with $0 < p < \frac{1}{2} < q < 1$. Because $\bar{f}(x)$ has only ordinary discontinuities, we can assume there is a value $\bar{d}(p,q) > 0$ such that $\bar{f}(x) > \bar{d}(p,q)$ for all x in $(\bar{F}^{-1}(p), \bar{F}^{-1}(q))$. We have

$$Y_{np_n+(j-1)L_n} = \bar{F}^{-1}\left(\frac{np_n+(j-1)L_n}{n} + \frac{\bar{W}_j(n)}{\sqrt{n}}\right) = \bar{F}^{-1}\left(\frac{np_n+(j-1)L_n}{n}\right) + \frac{\bar{W}_j(n)}{\sqrt{n}\bar{f}(y^*(j,n))}$$

where $y^*(j,n)$ is between $\frac{np_n+(j-1)L_n}{n}$ and $\frac{np_n+(j-1)L_n}{n} + \frac{\bar{W}_j(n)}{\sqrt{n}}$. It follows that there exists a value $d^*(p,q) < \infty$ such that

$$\lim_{n\to\infty} P \left[\begin{array}{l} \left| Y_{np_n+(j-1)L_n} - \bar{F}^{-1}\left(\dfrac{np_n+(j-1)L_n}{n}\right) \right| < \dfrac{d^*(p,q)}{\sqrt{n}} \\ \\ \text{for all } j \text{ such that } np \leqq np_n+(j-1)L_n \leqq nq \end{array} \right] = 1.$$

Let $J_1^{(n)}(p,q)$ denote the smallest integer j satisfying $np \leqq np_n + (j-1)L_n$, and $J_2^{(n)}(p,q)$ denote the largest integer j satisfying $np_n + (j-1)L_n \leqq nq$. We have

$$W_n(\hat{\theta}_1(n), \hat{\theta}_2(n)) >$$

$$\frac{n(L_n-1)}{L_n^2}\left(\frac{n}{(\hat{\theta}_2(n))^2}\right) \sum_{j=J_1^{(n)}(p,q)}^{J_2^{(n)}(p,q)} \left\{ \begin{array}{l} u(j+1,n)\ (Y_{np_n+jL_n} - \hat{\theta}_1(n) - \hat{\theta}_2(n)U(j+1,n)) \\ \\ - u(j,n)\ (Y_{np_n+(j-1)L_n} - \hat{\theta}_1(n) - \hat{\theta}_2(n)U(j,n)) \end{array} \right\}^2$$

Using the facts developed above, we can write the sum in this last expression as

$$\frac{L_n}{n}\left\{\int_p^q \left[\frac{d}{dx}\{g(G^{-1}(x))[\bar{F}^{-1}(x) - \bar{\theta}_1 - \bar{\theta}_2 G^{-1}(x)]\}\right]^2 dx + \epsilon(n, p, q)\right\}$$

where $\epsilon(n, p, q)$ converges stochastically to zero as n increases.

Define $I(p, q)$ as $\int_p^q \left[\frac{d}{dx}\{g(G^{-1}(x))[\bar{F}^{-1}(x) - \bar{\theta}_1 - \bar{\theta}_2 G^{-1}(x)]\}\right]^2 dx$. $I(p, q)$ is zero if and only if $g(G^{-1}(x))[\bar{F}^{-1}(x) - \bar{\theta}_1 - \bar{\theta}_2 G^{-1}(x)]$ is equal to some constant Γ for all x in (p, q). Since $\bar{F}^{-1}\left(\frac{1}{2}\right) - \bar{\theta}_1 - \bar{\theta}_2 G^{-1}\left(\frac{1}{2}\right) = 0$, Γ must be zero, and $\bar{F}^{-1}(x) = \bar{\theta}_1 + \bar{\theta}_2 G^{-1}(x)$ for all x in (p, q). Since under $\bar{F}(x)$ the hypothesis is false, there must be some p, q with $0 < p < 1/2 < q < 1$, such that $I(p, q) > 0$. For such p, q the probability approaches one as n increases that $W_n(\hat{\theta}_1(n), \hat{\theta}_2(n)) > \frac{n}{2\bar{\theta}_2^2} I(p, q)$ and this last expression becomes much larger than $(K_n + 1) + \sqrt{2(K_n + 1)}\, t(\alpha)$. This proves consistency for this last case, where $\bar{f}(x)$ is positive for all x in $(\bar{F}^{-1}(0), \bar{F}^{-1}(1))$.

4. CONCLUDING REMARKS

There are many test procedures for testing the hypothesis described above. The best known are chi-square, Kolmogorov-Smirnov, and Cramér-Von Mises type tests. These are described in References [1]–[4]. The choice among these tests would be made on the basis of their powers against alternatives which are not too far from the hypothesis. The power of a test can be found either by Monte Carlo or analytic methods, and References [1]–[4] give some partial results along these lines. Under some alternatives which approach the hypothesis as n increases, it is possible to compute the asymptotic power of the test based on $W_n(\hat{\theta}_1(n), \hat{\theta}_2(n))$, but the computations are lengthy.

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EXACT FIRST AND SECOND ORDER MOMENTS OF ORDER STATISTICS FROM THE TRUNCATED EXPONENTIAL DISTRIBUTION

A. K. Md. Ehsanes Saleh, Christine Scott, and D. Bruce Junkins

Carleton University
Ottawa, Canada

ABSTRACT

Exact expressions for the first and second order moments of order statistics from the truncated exponential distribution, when the proportion $1-P$ of truncation is known in advance, are presented in this paper. Tables of expected values and variances-covariances are given for $P=0.5\ (0.1)\ 0.9$ and $n=1\ (1)\ 10$.

1. INTRODUCTION

The exponential distribution has been in considerable use both in truncated and untruncated form as a model in the analysis of life-testing data and often the moments of first order and second order of order statistics have considerable value when applied to the problem of linear estimation and test of hypothesis. The works of Epstein [1, 2], Epstein and Sobel [3], Gumbel [4], Sarhan [5, 6] and Sarhan and Greenberg [7] culminated in the publication of expected values and variances-covariances of order statistics from untruncated exponential distribution. The use of these results in the estimation problem has also been considered by the same above authors. We may mention the Savage test [8] for scale parameter for densities defined on the positive half of the real line which make use of the expected values of order statistics from untruncated exponential distribution.

In this paper, we present the exact expressions for the first and second order moments of order statistics from a truncated exponential distribution when the proportion $(1-P)$ of truncation is known in advance. In addition, we provide tabular values of first moments and variance-covariances of order statistics for $n=1\ (1)\ 10$ and $P=0.5\ (0.1)\ 0.9$. We may mention the work of Tarter (1966) who considered the truncated logistic distribution.

2. MOMENTS OF ORDER STATISTICS FROM TRUNCATED EXPONENTIAL DISTRIBUTION

Suppose we are sampling from the right truncated exponential distribution given by

$$F^*(x)=\frac{1-\exp(-x)}{P}\ ; \qquad 0\leq x\leq -\ln(1-P), \tag{2.1}$$

where $1-P$ is the proportion of truncation on the right of the exponential distribution

(2.2) $$F(x)=1-\exp(-x); \qquad 0\leqslant x\leqslant\infty.$$

The quantity P is known in advance. Let

$$X_{(1,n)}<X_{(2,n)}\leqslant \ldots . \leqslant X_{(n,n)}$$

be an ordered sample of size n from (2.1). Thus, $X_{(k,n)}$ is the order statistic with rank k in a sample of size n. Our problem is to obtain expressions for the first and second order moments of order statistics.

From (2.1) it is clear that the inverse of $y=F^*(x)$ is

(2.3) $$F^{*-1}(y)=-\ln(1-Py).$$

Then, the Jth order ordinary moment of the Kth order statistic $X_{(k,n)}$ from a sample size n, $E(X^J_{(k,n)})$ is given by

(2.4) $$E(X^J_{(k,n)})=n\cdot\binom{n-1}{k-1}\int_0^1[-\ln(1-Py)]^J y^{k-1}(1-y)^{n-k}dy.$$

For $J=1$ the above expression reduces to

(2.5) $$E(X_{(k,n)})=n\cdot\binom{n-1}{k-1}\int_0^1[-\ln(1-Py)]y^{k-1}(1-y)^{n-k}dy$$

$$=n\cdot\binom{n-1}{k-1}\sum_{j=1}^{\infty}\frac{P^j}{j}\int_0^1 y^{k+j-1}(1-y)^{n-k}\,dy.$$

Setting $$-\ln(1-Py)=\sum_{j=1}^{\infty}\frac{(Py)^j}{j}$$

The expression at (2.5) may be written as

(2.6)

$$n\cdot\binom{n-1}{k-1}\sum_{j=1}^{\infty}\frac{P^j}{j}\sum_{r=0}^{n-k}(-1)^r\binom{n-k}{r}\frac{1}{k+r+j}=n\cdot\binom{n-1}{k-1}\sum_{r=0}^{n-k}(-1)^r\binom{n-k}{r}\sum_{j=1}^{\infty}\frac{P^j}{j(k+r+j)}$$

$$=n\cdot\binom{n-1}{k-1}\sum_{r=0}^{n-k}(-1)^r\binom{n-k}{r}\frac{1}{k+r}\left[-\ln(1-P)-\frac{1}{P^{k+r}}\left\{-\ln(1-P)-\sum_{i=1}^{k+r}\frac{P^\nu}{\nu}\right\}\right].$$

For $k=n$, we obtain from (2.6)

(2.7) $$E(X_{(n,n)})=-\ln(1-P)-\frac{1}{P^n}\left(-\ln(1-P)-\sum_{\nu=1}^{n}\frac{P^\nu}{\nu}\right).$$

Thus, we have the following recurrence relation which is useful for computational purposes—

$$E(X_{(k,n)}) = n \sum_{r=0}^{n-k} (-1)^r \binom{n-k}{r} \binom{n-1}{k-1} \frac{E(X_{(k+r,k+r)})}{k+r}. \tag{2.8}$$

For example

$$E(X_{(1,n)}) = \sum_{r=1}^{n} (-1)^{r+1} \binom{n}{r} E(X_{(r,r)}). \tag{2.9}$$

We note that the expected values of $X_{(k,n)}$ are related to the expected values of extreme order statistics $X_{(k+r,k+r)}$. Once $X_{(k+r,k+r)}$ are known the others may be calculated by using (2.8).

A table of the expected values of the order statistics has been prepared for $n = 1(1)10$ and $P = 0.5(0.1)0.9$ using formula (2.6) and the recurrence formula (2.8) has been checked by numerical integration for accuracy. The two results agree up to the eight decimal places on which the computation has been based; however, tabular values are reported for only four decimal places.

For $J = 2$ the expression at (2.4) reduces to

$$E(X^2_{(k,n)}) = n \cdot \binom{n-1}{k-1} \int_0^1 -\ln^2(1-Py)\, y^{k-1} (1-y)^{n-k}\, dy. \tag{2.10}$$

The right-hand side may be written as

$$n \cdot \binom{n-1}{k-1} \sum_{i=1}^{\infty} \frac{P^i}{i} \int_0^1 y^{k+i-1} (1-y)^{n-k} [-\ln(1-Py)]\, dy. \tag{2.11}$$

The integral in (2.11) is similar to the integral appearing in $E(X_{(k,n)})$ and we may write (2.11) as

(2.12)

$$E(X^2_{(k,n)}{}^s) = n \cdot \binom{n-1}{k-1} \sum_{r=0}^{n-k} (-1)^r \binom{n-k}{r} \sum_{i=1}^{\infty} \frac{P^i}{i(k+r+i)} \left[-\ln(1-P) - \frac{1}{P(k+r+i)} \left\{-\ln(1-P) - \sum_{\nu=1}^{k+r+i} \frac{P^\nu}{\nu}\right\}\right].$$

The above expression may finally be written as

(2.13)

$$E(X^2_{(k,n)}) = -\ln(1-P) E(X_{(k,n)}) - n \cdot \binom{n-1}{k-1} \sum_{r=0}^{n-k} (-1)^r \binom{n-k}{r} \left[\frac{2G_{k+r}}{k+r} \sum_{i=1}^{k+r} \frac{1}{i} - \frac{1}{k+r} \sum_{i=1}^{k+r} \frac{G_{k+r-i}}{i}\right]$$

by letting

$$G_s=\frac{1}{P^s}\sum_{\nu=s+1}^{\infty}\frac{P^\nu}{\nu}=\frac{1}{P^s}\left\{-\ln(1-P)-\sum_{\nu=1}^{s}\frac{P^\nu}{\nu}\right\} \qquad \text{if } s>0,$$

$$=-\ln(1-P) \qquad \text{if } s=0,$$

and few manipulative steps to convert the infinite sum in finite form.

For $k=n$, the expression (2.13) reduces to

$$E(X^2_{(n,n)})=-\ln(1-P)E(X_{(n,n)})-\left[2G_n\sum_{i=1}^{n}\frac{1}{i}-\sum_{i=1}^{n}\frac{G_{n-i}}{i}\right]. \tag{2.14}$$

This gives us the following recurrence relation

$$E(X^2_{(k,n)})=-\ln(1-P)E(X_{(k,n)})-n\sum_{r=0}^{n-k}(-1)^r \frac{\binom{n-k}{r}\binom{n-1}{k-1}}{k+r}[-\ln(1-P)E(X_{(k+r,k+r)})-E(X^2_{(k+r,k+r)})]. \tag{2.15}$$

For example, taking $k=1$, we obtain

$$E(X^2_{(1,n)})=-\ln(1-P)E(X_{(1,n)})-\sum_{r=1}^{n}(-1)^{r+1}\binom{n}{r} [-\ln(1-P)\cdot E(X_{(r,r)})-E(X^2_{(r,r)})]. \tag{2.16}$$

Therefore, we can compute $E(X^2_{(k,n)})$ based on the expected values of $X_{(k,n)}$ and $X^2_{(r,r)}$ for $r=k, k+1, \ldots, n$. The variance of the order statistic may easily be calculated from the above result. Tables of variances have been obtained for $n=1\,(1)\,10$ and $P=0.5\,(0.1)\,0.9$ with the use of formula (2.13). The recurrence formula (2.15) has been checked for accuracy by numerical integration. The two results agree up to eight decimal places on which the computation is based. We report values only for four decimal places.

3. PRODUCT MOMENTS OF ORDER STATISTICS

The product moment of order 2 of the order statistics from truncated exponential distribution (2.1) is given by

$$E[X_{(k,n)}X_{(l,n)}]=C_{kl}\int_0^1[-\ln(1-Py_l)^{n-l}\int_0^{y_l}[-\ln(1-Py_k)\cdot y_k^{k-1}(y_l-y_k)^{l-k-1}dy_k\,dy_l \tag{3.1}$$

for $k<l$ where

$$c_{kl}=\frac{n!}{(n-l)!(k-1)!(l-k-1)!}$$

by using Sarhan and Greenberg [7].

We evaluate the integral using the same techniques as have been used to evaluate $E(X_{(k,n)})$ and $E(X^2_{(k,n)})$ and using the method of Tarter [9]. Thus,

(3.2)
$$E[X_{(k,n)}X_{(l,n)}] = c_{kl}\sum_{i=1}^{\infty}\frac{P^i}{i}\int_0^1[-\ln(1-Py_l)]\cdot(1-y_l)^{n-l}\int_0^{y_l}y_k^{k+i-1}(y_l-y_k)^{l-k-1}dy_k dy_l$$

(3.3)
$$= c_{kl}\sum_{s=0}^{l-k-1}(-1)^s\binom{l-k-1}{s}\sum_{i=1}^{\infty}\frac{P^i}{i(k+s+i)}\int_0^1[-\ln(Py_l)](1-y_l)^{n-1}y_l^{l+i-1}dy_l$$
$$= c_{kl}\sum_{r=0}^{n-l}(-1)^r\binom{n-l}{r}\sum_{s=0}^{l-k-1}(-1)^s\binom{l-k-1}{s}\sum_{i=1}^{\infty}\frac{P^i}{i(k+s+i)(l+i+r)}\left(\ln(1-P)^{-1}\right.$$
$$\left.-\frac{1}{P^{l+r+i}}\left(\ln(1-P)^{-1}-\sum_{\nu=1}^{l+r+i}\frac{P^\nu}{\nu}\right)\right)$$

(3.4)
$$= c_{kl}\sum_{r=0}^{n-l}(-1)^r\binom{n-l}{r}\sum_{s=0}^{l-k-1}(-1)^s\binom{l-k-1}{s}[T_1+T_2],$$

where

(3.5)
$$T_1=\frac{1}{l+r-k-s}\ln(1-P)^{-1}\left[\frac{1}{k+s}(n(1-P)^{-1}-G_{k+s})-\frac{1}{l+r}(\ln(1-P)^{-1}-G_{l+r})\right]$$
$$-\frac{1}{l+r-k-s}G_{l+r}\left[\frac{1}{k+s}\sum_{i=1}^{k+s}\frac{1}{i}-\frac{1}{l+r}\sum_{i=1}^{l+r}\frac{1}{i}\right],$$

and

(3.6)
$$T_2=\frac{1}{(k+s)(l+r-k-s)}\sum_{i=1}^{k+s}\frac{1}{l+r-i+1}(G_{i-1}-G_{l+r})-\frac{1}{(l+r)(l+r-k-s)}\sum_{i=1}^{l+r}\frac{1}{i}(G_{l+r-i}-G_{l+r}).$$

Substituting (3.5) and (3.6) in (3.4) gives

$$E[X_{(k,n)}X_{(l,n)}] = c_{kl}\sum_{r=0}^{n-l}(-1)^r\binom{n-l}{r}\sum_{s=0}^{l-k-1}(-1)^s\binom{l-k-1}{s}\frac{1}{l+r-k-s}$$
$$\left[\ln(1-P)^{-1}\left(\frac{1}{k+s}(\ln(1-P)^{-1}-G_{k+s})-\frac{1}{l+r}(\ln(1-P)^{-1}-G_{l+r})\right)\right.$$
$$-G_{l+r}\left(\frac{1}{k+s}\sum_{i=1}^{k+s}\frac{1}{i}-\frac{1}{l+r}\sum_{i=1}^{l+r}\frac{1}{i}\right)$$
$$\left.+\frac{1}{k+s}\sum_{i=1}^{k+s}\frac{1}{l+r-i+1}(G_{i-1}-G_{l+r})-\frac{1}{l+r}\sum_{i=1}^{l+r}\frac{1}{i}(G_{l+r-i}-G_{l+r})\right],$$

or

$$E[X_{(k,n)}X_{(l,n)}]=c_{kl}\sum_{r=0}^{n-l}(-1)^r\binom{n-l}{r}\sum_{s=0}^{l-k-1}(-1)^s\binom{l-k-1}{s}\frac{1}{l+r-k-s}$$
$$\left[\ln(1-P)^{-1}\left(\frac{1}{k+s}E[X_{(k+s,k+s)}]-\frac{1}{l+r}E[X_{(l+r,l+r)}]\right)\right.$$
$$-G_{l+r}\left(\frac{1}{k+s}\sum_{i=1}^{k+s}\left(\frac{1}{i}+\frac{1}{l+r-i+1}\right)-\frac{1}{l+r}\sum_{i=1}^{l+r}\frac{2}{i}\right)$$
$$\left.+\frac{1}{k+s}\sum_{i=1}^{k+s}\frac{1}{l+r-i+1}G_{i-1}-\frac{1}{l+r}\sum_{i=1}^{l+r}\frac{1}{i}G_{l+r-i}\right].$$

It is clear from the final expression that the product moments are also calculable using the expected values of $X_{(k,n)}$ and the expected values of the squares of extreme order statistics. By using these, the tables of covariances have been computed for $n=2\ (1)\ 10$ and $P=0.5\ (0.1)\ 0.9$ and checked by numerical integration. The two results agree up to the eight decimal places on which it is based; however, we report only four decimal places.

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TABLE 1. *Table of expected values of order statistics from truncated exponential distribution for* n = 1 (1) 10 *and* P = 0.5 (0.1) 0.9

$p = 0.5$										
n \ k	1	2	3	4	5	6	7	8	9	10
1	0.3069									
2	0.1931	0.4206								
3	0.1402	0.2991	0.4813							
4	0.1098	0.2313	0.3668	0.5195						
5	0.0902	0.1883	0.2958	0.4142	0.5458					
6	0.0765	0.1587	0.2475	0.3440	0.4493	0.5651				
7	0.0664	0.1371	0.2127	0.2940	0.3815	0.4764	0.5798			
8	0.0586	0.1206	0.1865	0.2565	0.3314	0.4116	0.4980	0.5915		
9	0.0525	0.1077	0.1659	0.2275	0.2928	0.3622	0.4363	0.5157	0.6010	
10	0.0475	0.0973	0.1495	0.2044	0.2622	0.3234	0.3881	0.4570	0.5303	0.6088
$p = 0.6$										
1	0.3891									
2	0.2406	0.5377								
3	0.1730	0.3758	0.6186							
4	0.1347	0.2877	0.4639	0.6702						
5	0.1102	0.2327	0.3702	0.5264	0.7062					
6	0.0932	0.1952	0.3077	0.4328	0.5732	0.7328				
7	0.0807	0.1681	0.2631	0.3671	0.4820	0.6097	0.7533			
8	0.0712	0.1475	0.2297	0.3187	0.4156	0.5218	0.6390	0.7696		
9	0.0637	0.1314	0.2038	0.2815	0.3652	0.4559	0.5547	0.6631	0.7829	
10	0.0576	0.1185	0.1831	0.2520	0.3257	0.4047	0.4900	0.5825	0.6833	0.7940
$p = 0.7$										
1	0.4840									
2	0.2926	0.6755								
3	0.2079	0.4618	0.7823							
4	0.1609	0.3492	0.5745	0.8516						
5	0.1311	0.2802	0.4526	0.6557	0.9005					
6	0.1105	0.2338	0.3730	0.5322	0.7174	0.9371				
7	0.0955	0.2005	0.3170	0.4476	0.5957	0.7661	0.9656			
8	0.0841	0.1755	0.2756	0.3861	0.5091	0.6476	0.8056	0.9885		
9	0.0751	0.1560	0.2437	0.3394	0.4444	0.5608	0.6911	0.8384	1.0073	
10	0.0678	0.1404	0.2184	0.3027	0.3943	0.4945	0.6050	0.7279	0.8660	1.0230
$p = 0.8$										
1	0.5976									
2	0.3506	0.8447								
3	0.2457	0.5604	0.9868							
4	0.1886	0.4170	0.7038	1.0812						
5	0.1529	0.3315	0.5453	0.8094	1.1491					
6	0.1285	0.2749	0.4447	0.6459	0.8912	1.2007				
7	0.1107	0.2347	0.3753	0.5372	0.7275	0.9567	1.2414			
8	0.0973	0.2048	0.3246	0.4598	0.6146	0.7952	1.0105	1.2744		
9	0.0868	0.1816	0.2859	0.4019	0.5322	0.6806	0.8525	1.0556	1.3017	
10	0.0783	0.1631	0.2555	0.3569	0.4693	0.5951	0.7376	0.9017	1.0941	1.3248
$p = 0.9$										
1	0.7442									
2	0.4173	1.0710								
3	0.2870	0.6780	1.2675							
4	0.2181	0.4935	0.8625	1.4025						
5	0.1758	0.3875	0.6525	1.0025	1.5025					
6	0.1471	0.3189	0.5247	0.7803	1.1136	1.5802				
7	0.1265	0.2709	0.4389	0.6392	0.8861	1.2046	1.6428			
8	0.1109	0.2355	0.3772	0.5416	0.7368	0.9757	1.2810	1.6945		
9	0.0988	0.2082	0.3308	0.4701	0.6311	0.8213	1.0529	1.3461	1.7381	
10	0.0890	0.1866	0.2946	0.4153	0.5522	0.7099	0.8956	1.1203	1.4026	1.7753

TABLE 2. *Table of variances and covariances of order statistics from truncated exponential distribution*

N	L/K	$P=0.50$									$N=1\ (1)\ 10$
		1	2	3	4	5	6	7	8	9	10
1	1	0.0391									
2	1	0.0026									
2	2	0.0129	0.0297								
3	1	0.0139									
3	2	0.0103	0.0231								
3	3	0.0058	0.0130	0.0220							
4	1	0.0093									
4	2	0.0076	0.0167								
4	3	0.0056	0.0123	0.0203							
4	4	0.0031	0.0068	0.0113	0.0167						
5	1	0.0066									
5	2	0.0057	0.0123								
5	3	0.0046	0.0100	0.0164							
5	4	0.0034	0.0073	0.0119	0.0174						
5	5	0.0019	0.0040	0.0065	0.0095	0.0131					
6	1	0.0049									
6	2	0.0044	0.0094								
6	3	0.0038	0.0081	0.0130							
6	4	0.0030	0.0065	0.0105	0.0151						
6	5	0.0022	0.0047	0.0076	0.0109	0.0148					
6	6	0.0012	0.0026	0.0041	0.0059	0.0080	0.0105				
7	1	0.0038									
7	2	0.0035	0.0073								
7	3	0.0031	0.0065	0.0104							
7	4	0.0026	0.0056	0.0089	0.0127						
7	5	0.0021	0.0045	0.0071	0.0102	0.0137					
7	6	0.0015	0.0032	0.0051	0.0073	0.0098	0.0127				
7	7	0.0008	0.0017	0.0028	0.0039	0.0053	0.0068	0.0086			
8	1	0.0030									
8	2	0.0028	0.0059								
8	3	0.0025	0.0053	0.0085							
8	4	0.0022	0.0047	0.0075	0.0106						
8	5	0.0019	0.0040	0.0064	0.0090	0.0120					
8	6	0.0015	0.0032	0.0051	0.0072	0.0096	0.0123				
8	7	0.0011	0.0023	0.0036	0.0051	0.0068	0.0087	0.0109			
8	8	0.0006	0.0012	0.0019	0.0028	0.0037	0.0047	0.0059	0.0072		
9	1	0.0025									
9	2	0.0023	0.0048								
9	3	0.0021	0.0044	0.0070							
9	4	0.0019	0.0040	0.0063	0.0089						
9	5	0.0017	0.0035	0.0056	0.0078	0.0103					
9	6	0.0014	0.0030	0.0047	0.0066	0.0087	0.0111				
9	7	0.0011	0.0024	0.0037	0.0053	0.0070	0.0089	0.0110			
9	8	0.0008	0.0017	0.0027	0.0037	0.0049	0.0063	0.0078	0.0095		
9	9	0.0004	0.0009	0.0014	0.0020	0.0026	0.0034	0.0042	0.0051	0.0061	
10	1	0.0020									
10	2	0.0019	0.0040								
10	3	0.0018	0.0037	0.0059							
10	4	0.0016	0.0034	0.0054	0.0075						
10	5	0.0015	0.0031	0.0048	0.0068	0.0089					
10	6	0.0013	0.0027	0.0042	0.0059	0.0078	0.0099				
10	7	0.0011	0.0023	0.0036	0.0050	0.0066	0.0083	0.0103			
10	8	0.0009	0.0018	0.0028	0.0040	0.0052	0.0066	0.0082	0.0099		
10	9	0.0006	0.0013	0.0020	0.0028	0.0037	0.0047	0.0058	0.0070	0.0083	
10	10	0.0003	0.0007	0.0011	0.0015	0.0020	0.0025	0.0031	0.0037	0.0044	0.0053
		$P=0.60$									$N=1\ (1)\ 10$

TABLE 2. *Table of variances and covariances of order statistics from truncated exponential distribution*—Continued

N	L/K	$P = 0.60$								$N = 1$ (1) 10	
		1	2	3	4	5	6	7	8	9	10
1	1	0.0671									
2	1	0.0370									
2	2	0.0221	0.0531								
3	1	0.0221									
3	2	0.0170	0.0393								
3	3	0.0099	0.0229	0.0404							
4	1	0.0145									
4	2	0.0122	0.0275								
4	3	0.0092	0.0208	0.0355							
4	4	0.0053	0.0119	0.0204	0.0314						
5	1	0.0102									
5	2	0.0090	0.0198								
5	3	0.0075	0.0165	0.0276							
5	4	0.0056	0.0124	0.0207	0.0310						
5	5	0.0032	0.0070	0.0117	0.0176	0.0250					
6	1	0.0075									
6	2	0.0068	0.0148								
6	3	0.0060	0.0130	0.0214							
6	4	0.0049	0.0107	0.0177	0.0261						
6	5	0.0037	0.0080	0.0131	0.0193	0.0269					
6	6	0.0020	0.0045	0.0073	0.0108	0.0151	0.0203				
7	1	0.0058									
7	2	0.0053	0.0115								
7	3	0.0048	0.0103	0.0168							
7	4	0.0042	0.0090	0.0146	0.0213						
7	5	0.0034	0.0074	0.0120	0.0175	0.0240					
7	6	0.0025	0.0054	0.0088	0.0129	0.0176	0.0234				
7	7	0.0014	0.0030	0.0049	0.0071	0.0098	0.0130	0.0169			
8	1	0.0045									
8	2	0.0043	0.0091								
8	3	0.0039	0.0084	0.0135							
8	4	0.0035	0.0075	0.0121	0.0174						
8	5	0.0030	0.0065	0.0105	0.0151	0.0205					
8	6	0.0025	0.0053	0.0086	0.0123	0.0167	0.0219				
8	7	0.0018	0.0039	0.0063	0.0090	0.0122	0.0160	0.0205			
8	8	0.0010	0.0021	0.0035	0.0050	0.0067	0.0088	0.0113	0.0143		
9	1	0.0037									
9	2	0.0035	0.0074								
9	3	0.0032	0.0069	0.0110							
9	4	0.0030	0.0063	0.0101	0.0144						
9	5	0.0027	0.0056	0.0090	0.0129	0.0173					
9	6	0.0023	0.0049	0.0078	0.0111	0.0149	0.0193				
9	7	0.0019	0.0040	0.0063	0.0090	0.0121	0.0157	0.0199			
9	8	0.0014	0.0029	0.0046	0.0066	0.0088	0.0114	0.0144	0.0180		
9	9	0.0007	0.0016	0.0025	0.0036	0.0048	0.0062	0.0079	0.0099	0.0122	
10	1	0.0030									
10	2	0.0029	0.0061								
10	3	0.0027	0.0058	0.0091							
10	4	0.0025	0.0053	0.0085	0.0121						
10	5	0.0023	0.0049	0.0078	0.0110	0.0147					
10	6	0.0021	0.0044	0.0069	0.0098	0.0131	0.0168				
10	7	0.0018	0.0037	0.0059	0.0084	0.0112	0.0144	0.0181			
10	8	0.0014	0.0030	0.0048	0.0068	0.0091	0.0117	0.0147	0.0181		
10	9	0.0010	0.0020	0.0035	0.0049	0.0066	0.0084	0.0106	0.0131	0.0160	
10	10	0.0006	0.0012	0.0019	0.0027	0.0036	0.0046	0.0058	0.0071	0.0087	0.0106
		$P = 0.70$								$N = 1$ (1) 10	

TABLE 2. *Table of variances and covariances of order statistics from truncated exponential distribution*—Continued

N	L/K	$P=0.70$					$N=1$ (1) 10				
		1	2	3	4	5	6	7	8	9	10
1	1	0.1125									
2	1	0.0583									
2	2	0.0367	0.0934								
3	1	0.0337									
3	2	0.0268	0.0647								
3	3	0.0164	0.0396	0.0735							
4	1	0.0216									
4	2	0.0187	0.0435								
4	3	0.0147	0.0340	0.0605							
4	4	0.0088	0.0204	0.0363	0.0586						
5	1	0.0149									
5	2	0.0134	0.0305								
5	3	0.0115	0.0261	0.0451							
5	4	0.0089	0.0202	0.0349	0.0543						
5	5	0.0053	0.0120	0.0206	0.0321	0.0477					
6	1	0.0109									
6	2	0.0100	0.0224								
6	3	0.0090	0.0200	0.0338							
6	4	0.0077	0.0170	0.0287	0.0436						
6	5	0.0059	0.0130	0.0220	0.0334	0.0482					
6	6	0.0034	0.0076	0.0128	0.0195	0.0281	0.0396				
7	1	0.0083									
7	2	0.0078	0.0170								
7	3	0.0071	0.0157	0.0260							
7	4	0.0064	0.0139	0.0232	0.0345						
7	5	0.0054	0.0118	0.0195	0.0291	0.0411					
7	6	0.0041	0.0089	0.0148	0.0221	0.0311	0.0428				
7	7	0.0024	0.0052	0.0086	0.0127	0.0180	0.0246	0.0334			
8	1	0.0065									
8	2	0.0062	0.0134								
8	3	0.0058	0.0125	0.0205							
8	4	0.0053	0.0114	0.0188	0.0276						
8	5	0.0047	0.0101	0.0166	0.0244	0.0339					
8	6	0.0039	0.0085	0.0139	0.0204	0.0284	0.0382				
8	7	0.0030	0.0064	0.0105	0.0154	0.0214	0.0287	0.0380			
8	8	0.0017	0.0037	0.0060	0.0088	0.0122	0.0164	0.0217	0.0285		
9	1	0.0052									
9	2	0.0050	0.0108								
9	3	0.0047	0.0102	0.0165							
9	4	0.0044	0.0095	0.0154	0.0224						
9	5	0.0040	0.0086	0.0140	0.0204	0.0279					
9	6	0.0035	0.0076	0.0124	0.0180	0.0246	0.0326				
9	7	0.0029	0.0063	0.0103	0.0150	0.0205	0.0272	0.0353			
9	8	0.0022	0.0048	0.0077	0.0112	0.0153	0.0203	0.0264	0.0340		
9	9	0.0013	0.0027	0.0044	0.0064	0.0087	0.0115	0.0150	0.0193	0.0247	
10	1	0.0043									
10	2	0.0041	0.0089								
10	3	0.0039	0.0084	0.0136							
10	4	0.0037	0.0079	0.0128	0.0184						
10	5	0.0035	0.0074	0.0119	0.0171	0.0232					
10	6	0.0031	0.0067	0.0108	0.0155	0.0211	0.0276				
10	7	0.0028	0.0059	0.0095	0.0136	0.0185	0.0242	0.0311			
10	8	0.0023	0.0049	0.0079	0.0113	0.0153	0.0201	0.0258	0.0326		
10	9	0.0017	0.0036	0.0058	0.0084	0.0114	0.0149	0.0191	0.0242	0.0305	
10	10	0.0010	0.0020	0.0033	0.0047	0.0064	0.0084	0.0108	0.0136	0.0172	0.0216
		$P=0.80$					$N=1$ (1) 10				

TABLE 2. *Table of variances and covariances of order statistics from truncated exponential distribution*—Continued

N	L/K	$P = 0.80$					$N = 1$ (1) 10				
		1	2	3	4	5	6	7	8	9	10
1	1	0.1905									
2	1	0.0907									
2	2	0.0610	0.1683								
3	1	0.0500									
3	2	0.0417	0.1062								
3	3	0.0273	0.0693	0.1387							
4	1	0.0311									
4	2	0.0279	0.0675								
4	3	0.0229	0.0554	0.1037							
4	4	0.0147	0.0354	0.0662	0.1147						
5	1	0.0211									
5	2	0.0196	0.0457								
5	3	0.0174	0.0406	0.0728							
5	4	0.0141	0.0329	0.0589	0.0965						
5	5	0.0089	0.0207	0.0369	0.0604	0.0962					
6	1	0.0152									
6	2	0.0144	0.0328								
6	3	0.0132	0.0302	0.0525							
6	4	0.0116	0.0265	0.0461	0.0728						
6	5	0.0093	0.0213	0.0369	0.0582	0.0883					
6	6	0.0058	0.0132	0.0228	0.0360	0.0544	0.0818				
7	1	0.0115									
7	2	0.0109	0.0245								
7	3	0.0103	0.0230	0.0392							
7	4	0.0094	0.0211	0.0359	0.0551						
7	5	0.0082	0.0184	0.0313	0.0481	0.0705					
7	6	0.0065	0.0146	0.0248	0.0381	0.0558	0.0804				
7	7	0.0040	0.0089	0.0152	0.0233	0.0340	0.0489	0.0704			
8	1	0.0089									
8	2	0.0086	0.0190								
8	3	0.0082	0.0181	0.0303							
8	4	0.0077	0.0169	0.0283	0.0427						
8	5	0.0070	0.0154	0.0258	0.0388	0.0556					
8	6	0.0061	0.0134	0.0224	0.0337	0.0482	0.0673				
8	7	0.0048	0.0105	0.0176	0.0265	0.0378	0.0528	0.0732			
8	8	0.0029	0.0064	0.0106	0.0160	0.0228	0.0318	0.0440	0.0613		
9	1	0.0072									
9	2	0.0069	0.0151								
9	3	0.0067	0.0145	0.0240							
9	4	0.0063	0.0138	0.0228	0.0338						
9	5	0.0059	0.0129	0.0213	0.0315	0.0443					
9	6	0.0053	0.0117	0.0193	0.0286	0.0401	0.0548				
9	7	0.0046	0.0101	0.0166	0.0246	0.0346	0.0472	0.0637			
9	8	0.0036	0.0079	0.0130	0.0192	0.0269	0.0368	0.0496	0.0667		
9	9	0.0022	0.0047	0.0078	0.0115	0.0161	0.0220	0.0296	0.0398	0.0539	
10	1	0.0059									
10	2	0.0057	0.0123								
10	3	0.0055	0.0119	0.0195							
10	4	0.0053	0.0114	0.0187	0.0274						
10	5	0.0050	0.0108	0.0177	0.0259	0.0359					
10	6	0.0046	0.0101	0.0164	0.0241	0.0333	0.0448				
10	7	0.0042	0.0091	0.0148	0.0217	0.0301	0.0404	0.0534			
10	8	0.0036	0.0078	0.0127	0.0186	0.0258	0.0346	0.0457	0.0600		
10	9	0.0028	0.0060	0.0099	0.0144	0.0199	0.0268	0.0353	0.0464	0.0610	
10	10	0.0017	0.0036	0.0058	0.0086	0.0118	0.0159	0.0209	0.0274	0.0360	0.0478
		$P = 0.90$					$N = 1$ (1) 10				

TABLE 2. *Table of variances and covariances of order statistics from truncated exponential distribution*—Continued

N	L/K	$P=0.90$					$N=1$ (1) 10				
		1	2	3	4	5	6	7	8	9	10
1	1	0.3454									
2	1	0.1433									
2	2	0.1068	0.3340								
3	1	0.0737									
3	2	0.0656	0.1804								
3	3	0.0477	0.1304	0.2949							
4	1	0.0441									
4	2	0.0414	0.1055								
4	3	0.0363	0.0924	0.1873							
4	4	0.0258	0.0655	0.1321	0.2579						
5	1	0.0292									
5	2	0.0280	0.0680								
5	3	0.0260	0.0631	0.1196							
5	4	0.0226	0.0547	0.1034	0.1834						
5	5	0.0158	0.0381	0.0718	0.1267	0.2266					
6	1	0.0207									
6	2	0.0201	0.0472								
6	3	0.0191	0.0449	0.0813							
6	4	0.0177	0.0415	0.0750	0.1251						
6	5	0.0152	0.0356	0.0643	0.1070	0.1756					
6	6	0.0104	0.0244	0.0439	0.0729	0.1192	0.2005				
7	1	0.0154									
7	2	0.0151	0.0345								
7	3	0.0145	0.0334	0.0585							
7	4	0.0138	0.0316	0.0555	0.0888						
7	5	0.0127	0.0290	0.0509	0.0813	0.1262					
7	6	0.0108	0.0247	0.0432	0.0690	0.1068	0.1663				
7	7	0.0073	0.0167	0.0291	0.0464	0.0717	0.1113	0.1788			
8	1	0.0119									
8	2	0.0117	0.0264								
8	3	0.0114	0.0257	0.0440							
8	4	0.0110	0.0247	0.0424	0.0658						
8	5	0.0104	0.0233	0.0400	0.0622	0.0928					
8	6	0.0095	0.0213	0.0365	0.0566	0.0844	0.1248				
8	7	0.0080	0.0179	0.0307	0.0476	0.0709	0.1047	0.1569			
8	8	0.0053	0.0120	0.0205	0.0317	0.0471	0.0694	0.1036	0.1605		
9	1	0.0095									
9	2	0.0093	0.0208								
9	3	0.0091	0.0203	0.0343							
9	4	0.0089	0.0197	0.0333	0.0506						
9	5	0.0085	0.0189	0.0319	0.0486	0.0705					
9	6	0.0080	0.0178	0.0300	0.0457	0.0663	0.0945				
9	7	0.0073	0.0162	0.0272	0.0414	0.0600	0.0855	0.1221			
9	8	0.0061	0.0135	0.0228	0.0346	0.0501	0.0713	0.1016	0.1477		
9	9	0.0040	0.0089	0.0150	0.0227	0.0329	0.0467	0.0665	0.0964	0.1451	
10	1	0.0077									
10	2	0.0076	0.0168								
10	3	0.0075	0.0165	0.0274							
10	4	0.0073	0.0161	0.0268	0.0401						
10	5	0.0071	0.0156	0.0259	0.0388	0.0552					
10	6	0.0068	0.0149	0.0248	0.0371	0.0528	0.0734				
10	7	0.0064	0.0140	0.0233	0.0348	0.0495	0.0687	0.0948			
10	8	0.0057	0.0126	0.0210	0.0314	0.0446	0.0619	0.0853	0.1186		
10	9	0.0048	0.0105	0.0174	0.0260	0.0370	0.0512	0.0706	0.0980	0.1390	
10	10	0.0031	0.0068	0.0114	0.0170	0.0240	0.0333	0.0458	0.0634	0.0898	0.1318

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AN OPERATOR THEORY OF PARAMETRIC PROGRAMMING FOR THE GENERALIZED TRANSPORTATION PROBLEM: I. BASIC THEORY*

V. Balachandran

Graduate School of Management
Northwestern University

and

G. L. Thompson

Graduate School of Industrial Administration
Carnegie-Mellon University

ABSTRACT

This paper gives a new organization of the theoretical results of the Generalized Transportation Problem with capacity constraints. A graph-theoretic approach is utilized to define the basis as a one-forest consisting of one-trees (a tree with an extra edge). Algorithmic development of the pivot-step is presented by the representation of a two-tree (a tree with two extra edges). Constructive procedures and proofs leading to an efficient computer code are provided. The basic definition of an operator theory which leads to the discussion of various operators is also given. In later papers we will present additional results on the operator theory for the generalized transportation problem based on the results in the present paper.

. INTRODUCTION

The Generalized Transportation Problem was introduced as an extension of the transportation roblem by A. R. Ferguson and G. B. Dantzig [9, 12] in their application to "The Problem of Routing ircraft." This was further applied by K. Eisemann and J. R. Lourie [11] for "The Machine Loading roblem," and also discussed and applied by A. Charnes and W. W. Cooper [8]. Formal methods f solving the generalized transportation problem and relevant theoretical insights are given in standard exts, for instance, A. Charnes and W. W. Cooper [8], G. B. Dantzig [9], W. W. Garwin [13] and G. adley [17]. Extensions of the loop-technique of the stepping-stone algorithm and related theoretical nderpinnings are presented by E. Balas and P. L. Ivanescu [6]. Results similar to those by Balas and vanescu [6], and consideration of degeneracy and upper bounds are presented by K. Eisemann [10] hile the topology of the generalized transportation problem at the end of each iteration of the stepping-tone method of solution was given by J. R. Lourie [19]; and two together developed the original version f the computer code [11] for IBM 704. Post-optimization and inclusion of additional constraints for

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the generalized transportation problem were treated by E. Balas [5], who utilized the dual-method and the poly-ω technique.

An operator theory of parametric programming for the transportation problem was developed by V. Srinivasan and G. L. Thompson in [21, 22]. In the present and subsequent [1, 2, 3] papers we extend the parametric approach to generalized transportation problems. The present paper develops the basic theoretical framework, including a new approach to representing one-trees (a tree with an extra edge) and two-trees (a tree with two extra edges) in a manner that is efficient for both theory and computation. The basic definitions of operators are also included. The detailed development of the various operators based on the results of the present paper is carried out in [1, 2, 3].

Throughout this paper, our emphasis is on constructive procedures and proofs since they lead directly to usable and hopefully efficient computer programs. Our development of operator theory for the generalized transportation problem is straight forward and requires no reference to the parametric programming theory for the general linear program [9, 20] for justification. Our aim in this paper is to provide simultaneously new theoretical insights and to give constructive procedure for obtaining the practical benefits of the new results. This also facilitates easy application and interpretation to certain management problems which are left for another paper.

The generalized transportation problem formulation arises in different contexts [9, 11, 12], but the most familiar application is the machine loading problem [11]. In this, *m types of machines* (rows) are available for the production of *n types of products* (columns). The production process is concerned with each unit of product being processed by a single machine and not by a specific sequence of machines. Each product may be produced by any one or more machines. The utilization of machine type i for product j requires e_{ij} hours per unit and costs c_{ij} dollars per unit. During a *fixed time period*, machines of type i have a maximum total capacity of a_i hours and product type j is required by an amount b_j. The machine loading problem is: In what amounts of x_{ij} should products be allotted to machines to attain production of required amounts within the available capacities of minimum total cost? Formulated as a linear programming model, the problem is:

$$\text{Minimize} \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij}$$

(1)
$$\text{Subject to:} \sum_{j=1}^{n} e_{ij}x_{ij} \leqslant a_i \qquad i=1, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} = b_j \qquad j=1, \ldots, n$$

$$x_{ij} \geqslant 0.$$

Due to similarity of (1) to the ordinary transportation problem and to take advantage of the structure in the constraint matrix, this is treated as a generalized transportation problem. In fact there can be weights similar to e_{ij} for the columns as well. For instance, consider the following problem:

$$\text{Minimize } \sum_{i=1}^{m}\sum_{j=1}^{n} c'_{ij}x'_{ij}$$

(2) $$\text{Subject to: } \sum_{j=1}^{n} a'_{ij}x'_{ij} \leqslant a_i; \qquad i=1, \ldots, m$$

$$\sum b'_{ij}x'_{ij} = b_j; \qquad j=1, \ldots, n$$

$$x'_{ij} \geqslant O,$$

where the a'_{ij}'s are nonnegative, a_i, b_j, b'_{ij} are positive and c'_{ij} are arbitrary. This problem (2) can be transformed to (1) (or with all column coefficients to be unity and arbitrary row coefficients or vice-versa). Without any loss of generality we shall transform the problem so that the column coefficients are unity as in (1). To do this we let $x_{ij}=b'_{ij}x'_{ij}$. Thus the coefficients for the row equations become $a'_{ij}/b'_{ij}=e_{ij}$ and the coefficients of the objective function become $c'_{ij}/b'_{ij}=c_{ij}$. Thus (2) and (1) are equivalent since $x'_{ij} \geqslant 0$ implies $x_{ij} \geqslant 0$ because b'_{ij} is positive.

Throughout the rest of this paper we shall concentrate on (1).

2. PRELIMINARY DEFINITIONS AND RESULTS

The inequalities given in (1) for the row sums can be changed to equalities by introducing slack variables $x_{i,n+1}$ for each i, with an associated weight $e_{i,n+1} \equiv 1$ and $c_{i,n+1} \equiv 0$ for all i. However no explicit constraint is imposed on the $(n+1)$st column sum (which can be interpreted as the total idle machine time). To ensure an initial feasible solution and also to ensure sufficient machine capacity, a fictitious row (machine type $m+1$) with very large machine capacity a_{m+1} is introduced. By assigning very high per unit production costs $c_{m+1,j}=M$ ($j \leqslant n$, and M a large positive number) any allotment to this (high cost) machine is penalized. In other words, if the optimal solution has a positive $x_{m+1,j}$ ($j \neq n+1$), then there is no feasible solution to the original problem. Also set $c_{m+1,n+1}=0$, $e_{m+1,n+1}=1$. If upper limits U_{ij} (capacity) for x_{ij} is imposed, then the *capacitated generalized transportation problem* can now be stated.

The following index sets are defined:

(3) $$I = \{1, 2, \ldots, m, m+1\} \text{ the set of machine types}$$

(3′) $$I' = \{1, 2, \ldots, m\} = I - \{(m+1)\}$$

(4) $$J = \{1, 2, \ldots, n, (n+1)\}: \text{ the set of product types}$$

(4′) $$J' = \{1, 2, \ldots, n\} = J - \{(n+1)\}$$

For $i \epsilon I$ and $j \epsilon J$, define the following quantities (each with its machine loading interpretation)

x_{ij}: amount of product type j to be allocated to machine type i.

c_{ij}: cost per unit of such allocation.

e_{ij}: production time of machine type i for product type j.

U_{ij}: maximum amount of product j that can be allocated to machine i. If there is no maximum then set $U_{ij} = N$.

a_i: availability (supply) of total time for machine type i.

b_j: requirement (demand) of product j.

We will refer to $c_{ij}, e_{ij}, U_{ij}, a_i$, and b_j as the *data* of the problem. In particular the a_i's and b_j's will be referred to as *rim conditions*. Thus the capacitated generalized transportation problem (P) will be:

$$\text{Minimize} \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} = Z, \tag{5}$$

subject to the following constraints

$$\sum_{j \in J} e_{ij} x_{ij} = a_i \qquad \text{for } i \in I \tag{6}$$

$$\sum_{i \in I} x_{ij} = b_j \qquad \text{for } j \in J' \tag{7}$$

$$0 \leq x_{ij} \leq U_{ij} \qquad \text{for } i \in I', \quad j \in J', \text{ and} \tag{8}$$

with the following assumptions

(A1) a_i, b_j, e_{ij}, U_{ij} are positive real numbers for all $i \in I'$, and $j \in J'$ and

(A2) the system (6–8) has a feasible solution (due to construction).

(A3) nondegenerate solutions exist. If degeneracy occurs (primal or dual) familiar methods [8, 10] could be used to prevent it.

REMARK 1: There is always a feasible solution to P due to construction of $(m+1)$st row. (A2). Finiteness of x_{ij} is due to constraint (8). Multiple optimal solutions if any are due to (dual) degeneracy [10]. $x_{m+1,j} > 0$ $(j \neq n+1)$ at the optimal solution is a sufficient condition that no feasible solution to the original problem (1) exists. We also discuss how to find an initial solution (which need not be optimal) later.

REMARK 2: For problem in which some or all of the x_{ij} are not bounded from above, we take $U_{ij} = N$ where N is a large positive number. If specific lower bounds are given viz $0 \leq l_{ij} \leq x_{ij} \leq U_{ij}$ for $i \in I'$ and $j \in J'$, it is possible to effect the linear transformation $x'_{ij} = x_{ij} - l_{ij}$ so that $0 \leq x'_{ij} \leq U'_{ij}$ where $U'_{ij} = (U_{ij} - l_{ij})$. This will change the rim conditions (6) and (7) as follows:

$$\sum_{i \in I} x'_{ij} = b'_j \qquad \text{for } j \in J' \text{ where } 0 \leq b'_j = b_j - \sum_{i \in I} l_{ij}, \text{ and} \tag{6'}$$

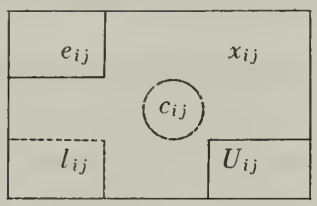
$$\sum_{j \in J} e_{ij} x'_{ij} = a'_i \qquad \text{for } i \in I \text{ where } 0 \leq a'_i = a_i - \sum_{j \in J} e_{ij} l_{ij}. \tag{7'}$$

Due to Assumption A2, there will be a feasible solution for x'_{ij} because the mapping is linear and preserves all conditions cited in (A1). The objective (5) will become

(5′) $$\sum_{i\in I}\sum_{j\in J} c_{ij}x'_{ij} + k \text{ where } k \text{ is a constant and equals } \sum_{i\in I}\sum_{j\in J} c_{ij}l_{ij}.$$

Since the transformation $X' = X + L$ is linear and (A1) and (A2) hold for the new problem, the optimal solution X^* for the original problem will correspond to optimal solution $X^{*\prime}$ where $X^{*\prime} = X^* - L = \{x^*_{ij} - l_{ij}\}$. Henceforth, we will tacitly assume that $l_{ij} \equiv 0$ for the generalized transportation problem.

By a *cell* (i, j) we mean an ordered index pair with row (machine type) $i\epsilon I$ and column (product type) $j\epsilon J$. We will use a tableau similar to that of the ordinary transportation problem. A cell of the tableau (matrix) contains the following information.



e_{ij}: in the north-west corner

x_{ij}: in the north-east corner (0 if left blank)

c_{ij}: in the center; if circled it implies cell (i, j) is in the basis

l_{ij}: in the southwest corner (if $l_{ij} = 0$ then it is *omitted;* if $l_{ij} > 0$ the transformation $x'_{ij} = x_{ij} - l_{ij}$ should be made as an initial step.)

U_{ij}: in the southeast corner = N if no upper bound is specified.

The following two propositions are valid.

PROPOSITION 1: The optimal solution will be unaltered if a constant cost $\delta_j (j = 1, 2, \ldots, n)$ is subtracted from (or added to) all the cost elements of a specified column, say $j = k$. $(j\epsilon J')$.

PROOF: Let the new costs be $c'_{ik} = c_{ik} - \delta_k$ (for kth col.) so that $c'_{ij} = c_{ij}, j \neq k$; $i\epsilon I$.

Then

$$\sum_{i\in I}\sum_{j\in J} c'_{ij}x_{ij} = \sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij} - \sum_{i\in I} \delta_k x_{ik}$$

$$= \sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij} - \delta_k \sum_{i\in I} x_{ik}$$

$$= \sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij} - b_k\delta_k.$$

Since $b_k\delta_k$ is constant, the optimal solution x_{ij} of problem (P) given by equations (5)–(8) will be optimal if c_{ij} is replaced by c'_{ij} $(i\epsilon I, j\epsilon J)$ in Equation (5); also the optimal solution found with costs c'_{ij} is also optimal with costs c_{ij}.

PROPOSITION 2: The optimal solution is unaltered if a cost $\delta_i e_{ij}$ is subtracted from (or added to) all the cost elements c_{ij} of a specified row $i=h, i\epsilon I'$.

PROOF: Let $c'_{hj}=c_{hj}-e_{hj}\delta_h$ (for hth row) and $c'_{ij}=c_{ij}$ for $i\neq h$ and $j\epsilon J$. Then

$$\sum_{i\epsilon I}\sum_{j\epsilon J} c'_{ij}x_{ij}=\sum_{i\epsilon I}\sum_{j\epsilon J} \qquad \sum_{j\epsilon J} e_{hj}x_{hj}$$

$$=\sum_{i\epsilon I}\sum_{j\epsilon J} c_{ij}x_{ij}-\delta_h a_h.$$

Since $\delta_h a_h$ is constant, the optimal solution of problem (P) given by equations (5)–(8) will be the same with costs being either c_{ij} or $c'_{ij}(i\epsilon I, j\epsilon J')$ in Equation (5).

We now discuss some basic concepts of graph theory. In what follows a *line* refers to a row or column of the tableau.

DEFINITION 1: Let Ω denote a collection of cells (i, j) of the tableau. *Line* g is connected to *line* h in Ω, if and only if there exists a *path* (sequence) S of distinct cells belonging to Ω.

(9) $$S=\{(i_1, j_1), (i_2, j_2) \ldots . (i_k, j_k)\}, \text{ such that}$$

(a) (i_1, j_1) is a cell of S in line g, and (i_k, j_k) is a cell of S in line h;
(b) for every $1<t<k$, either $i_{t-1}=i_t\neq i_{t+1}$ and $j_{t-1}\neq j_t=j_{t+1}$, or $i_{t-1}\neq i_t=i_{t+1}$ and $j_{t-1}=j_t\neq j_{t+1}$;
(c) If $j_t=n+1$, then $t=k$ or $t=1$ but not both;
(d) If the number of distinct lines between g and h, excluding g and h in the path S is r, then the *length* of the path connecting lines g and h is $l(g, h)=r+1$. If no path exists then $l(g, h)=\infty$.

REMARK 3: The above definition ensures a sequence of cells forming an alternating path, alternating between columns and rows or vice versa along connected lines.

DEFINITION 2: A *cycle* of order l is a path C containing $2l$ cells ($l\geq 2$), such that

(i) Every line of the matrix contains either 0 or 2 cells of C;
(ii) $C=S_\alpha\cup S_\beta$ where each set S_α and S_β contains l cells, such that
(a) if a line has two cells of C, then one belongs to S_α and the other to S_β, and
(b) $\alpha=\prod_{(i,j)\epsilon S_\alpha} e_{ij} > \prod_{(i,j)\epsilon S_\beta} e_{ij}=\beta$.

DEFINITION 3: The *circulation factor* of a cycle C is $\rho=\alpha/(\alpha-\beta)$.

REMARK 4; Sometimes, while pivoting, a closed path is created which satisfies all the requirements of Definition 2, except (ii) (b); i.e., we have $\alpha=\beta$ for the closed path. In this case it is possible to alter x_{ij} around the path and remove a cell thus breaking the path. (See the discussion of symmetric loops in Balas and Ivanescu [6], or Eisemann [10].) Moreover, no slack cell, i.e., a cell in column $(n+1)$, can be a member of a cycle by condition (c) of Definition 1.

DEFINITION 4: A *loop* is defined to be a slack cell $(i, n+1)$. We define the circulation factor of a loop to be infinity.

DEFINITION 5: Let Ω be a set of cells. The *span* $S(\Omega)$ is the set of all lines excluding column $n+1$) that contain at least one cell in Ω. We say that Ω is a *connected set* if every pair of lines in the pan $S(\Omega)$ is connected.

DEFINITION 6: A *one-tree* is a maximal connected set of cells such that either

(i) it is acyclic (does not contain a cycle) and there is exactly one cell from column $n+1$; i.e., it contains a loop, or

(ii) it contains no cell from column $n+1$, and has a unique cycle.

DEFINITION 7: A *basis B* for problem P is a *one-forest;* a set of $(m+n+1)$ cells (called basis cells) which is the union of one-trees T_j, $j=1, \ldots, k(2 \leq k < m+n+1)$, such that

(i) $S(T_i) \cap S(T_j) = \emptyset$ if $i \neq j$ and

(ii) $\bigcup_{i=1}^{k} S(T_i) = I \cup J'$.

REMARK 5: The $m+n+1$ cells forming a basis are identified in the tableau (Figure 3) by those cells where c_{ij}'s are circled.

DEFINITION 8: A *basic solution* $X=\{x_{ij}\}$ corresponding to a basis B satisfies (6) and (7) and is such that $(i,j) \notin B$ implies that either $x_{ij}=0$ or $x_{ij}=U_{ij}$. If in addition (8) is satisfied for $(i,j) \in B$, the basic solution is called *primal feasible.* We define LB and UB as the sets of nonbasic variables that are at their lower or upper bounds, respectively. We define (B, LB, UB) as a *basis structure.* It is known [6, 8] that given a problem and its basis structure, the associated primal solution is unique.

i ↓ MACH. \ j → PROD	1	2	3	4	a_i
1	3 / 3 / 0 / 15	2 / 6 / 0 / N	2 / 3 / 0 / 40	3 / 4 / 0 / 80	≤ 300
2	1 / 6 / 150	3 / 5 / 0 / 75	5 / 8 / 0 / 25	10 / 15 / 0 / N	≤ 225
3	4 / 1 / 0 / 60	2 / 3 / 0 / N	2 / 10 / 0 / 30	1 / 6 / 0 / 50	≤ 140
b_j	" 170	" 60	" 35	" 60	

FIGURE 1.

DEFINITION 9: A solution is said to be *primal nondegenerate* if $0 < x_{ij} < U_{ij}$ for $(i,j) \in B$.

DEFINITION 10: The cell $(m+1, n+1)$ is called the *absorbing cell.*

REMARK 6: It is shown by Eisemann [10] that the absorbing cell is always in the basis.

EXAMPLE: Consider a machine loading problem with the data given in a tableau form of Figure 1. In this, there are three (m) machine types and four (n) product types. The rows correspond to machine types with available time units of 300, 225, and 140, respectively. Product types correspond to columns

i \ j	1	2	3	4	5	a_i
1	3 | 3 | 15	2 | 6 | N	2 | 3 | 40	3 | 4 | 80	1 | 0 | N	300
2	1 | 6 | 150	3 | 5 | 75	5 | 8 | 25	10 | 15 | N	1 | 0 | N	225
3	4 | 1 | 60	2 | 3 | N	2 | 10 | 30	1 | 6 | 50	1 | 0 | N	140
4	1 | 100 | N	1 | 100 | N	1 | 100 | N	1 | 100 | N	1 | 0 | N	1000
b_j	170	60	35	60	–	

FIGURE 2.

and they are required in an amount of 170, 60, 35, and 60, respectively. The utilization of machine type i for product j requires e_{ij} time units per unit of product and is given in the northwest corner of the cell (i, j). The cost c_{ij}, the lower bound l_{ij} and upper bound U_{ij} are given in the center, southwest corner and southeast corner, respectively, by the cell (i, j). These are considered as the *data* of the problem. Figure 2 gives the same problem of Figure 1, but with a slack column 5 with $e_{i5}=1$, $c_{i5}=0=l_{i5}$ and $U_{i4}=N$ a large positive number for $i=1, 2, 3$ and an extra machine type, (row 4) introduced where $e_{4j}=1$; $l_{4j}=0$, $U_{4j}=N$ and $c_{4j}=100$, a relatively large positive cost for $j=1$ to 4. A large positive time unit, 1,000, is chosen for a_4. No value for b_5 is assigned since there is no constraint corresponding to the slack column.

Solution techniques for solving problem P are discussed in Balas and Ivnaescu [6], Eisemann [10] and in other text books [8, 9, 13]; however, the following algorithm for finding an intial primal fesaible solution has been found to be efficient.

ALGORITHM A1: For finding an initial feasible solution $X=\{x_{ij}\}$ and an initial basis structure (B, LB, UB).

(0) Start with all $x_{ij}=0$ and all cells $(i, j)\epsilon LB$. Let $SR=I$ and $SC=J'$. Let $j=1$.

(1) If $SC=\phi$ go to 7. Else go to 2.

(2) If $j=n+1$ set $j=1$; if $j\notin SC$ set $j=j+1$ and go to 2. Else go to 3.

(3) Find h such that $c_{hj}=\min_{i\epsilon SR} c_{ij}$ and $x_{ij}<U_{ij}$. Let $x_{hj}=\min\left\{\frac{a_h}{e_{hj}}, b_j, U_{hj}\right\}$.

(4) Let $a_h=a_h-e_{hj}x_{hj}$ and $b_j=b_j-x_{hj}$. If $a_h=b_j=0$ go to 6. Else go to 5.

(5) (a) If $b_j=0$, let $SC=SC-\{j\}$; $(h, j)\epsilon B$, $j=j+1$. Go to 1.
(b) If $a_h=0$; let $SR=SR-\{h\}$; $(h, j)\epsilon B$ and $j=j+1$. Go to 1.
(c) If $x_{hj}=U_{hj}$; let $(h, j)\epsilon UB$, $j=j+1$ and go to 1.

(6) Here $a_h=b_j=0$. If $|SR|<2$ (number of elements in sets SR) let $SC=SC-\{j\}$, $(h, j)\epsilon B$, $j=j+1$, and go to 1. Else (i.e., $|SR|\geq 2$), find $|SC|$. If $|SC|\geq 2$, let $SC=SC-\{j\}$, $(h, j)\epsilon B$, $j=j+1$, and go to 1. Else (i.e., $|SC|<2$) let $SR=SR-\{h\}$, $(h, j)\epsilon B$, $j=j+1$, and go to 1.

(7) For each $i\epsilon SR$, set $x_{i,n+1}=a_i$ and $(i, n+1)\epsilon B$. STOP.

u_i \ v_j	100	287	3	4		a_i
0	3 | 15 | 3 | 15	2 | 6 | N	2 | 35 | (3) | 40	3 | 60 | (4) | 80	1 | 5 | (0) | N	300
-94	1 | 45 | (6) | 150	3 | 60 | (5) | 75	5 | 8 | 25	10 | 15 | N	1 | 0 | N	225
-99/4	4 | 35 | (1) | 60	2 | 3 | N	2 | 10 | 30	1 | 6 | 50	1 | 0 | N	140
0	1 | 75 | (100) | N	1 | 100 | N	1 | 100 | N	1 | 100 | N	1 | 925 | (0) | N	1000
b_j	170	60	35	60	–	

FIGURE 3.

Figure 3 gives a starting primal feasible solution for the problem given in Figure 2. (Ignore, for the present the numbers given as row and column headings.) This solution is obtained by applying algorithm A1. The following give the steps of A1 and the results due to application of these steps sequentially.

Step No. *Result due to this step.*

(0) $x_{ij}=0$ and $(i, j)\epsilon LB$ for all (i, j).
$SR=\{1, 2, 3, 4\}$; $SC=\{1, 2, 3, 4\}$. $j=1$.

(2) $j\epsilon SC$. Go to 3.
(3) c_{31} is minimum. $x_{31}=35=a_3/e_{31}$.
(4) $a_3=0$; $b_1=135$.
(5) $SR=SR-\{3\}$; $j=2$; $(3, 1)\epsilon B$. Go to 1.
(3) c_{22} is minimum and $x_{22}=60=b_2$.
(4) $a_2=45$, $b_2=0$.
(5) $SC=SC-\{2\}$, $j=3$; $(2, 2)\epsilon B$. Go to 1.
(3) c_{13} is minimum. $x_{13}=35=b_3$.
(4) $a_1=230$; $b_3=0$.
(6) $SC=SC-\{3\}$; $j=4$; $(1, 3)\epsilon B$, go to 1.
(3) c_{14} is minimum. $x_{14}=60=b_4$.
(4) $a_1=50$; $b_4=0$.
(5) $SC=SC-\{4\}$; $j=5$; $(1, 4)\epsilon B$; go to 1.
(2) $j=5$ so $j=1$; go to 3.
(3) c_{11} is minimum. $x_{11}=U_{11}=15$.
(4) $a_1=5$; $b_1=120$.
(5) $(1, 1)\epsilon UB$; $j=2$, go to 1.
(2) $j=2$; $j=3$; $j=4$; $j=5$; so $j=1$. Go to 3.
(3) $x_{21}=45=a_2/e_{21}$.
(4) $a_2=0$; $b_1=75$.
(5) $SR=SR-\{2\}$, $j=2$; $(2, 1)\epsilon B$; go to 1.

(2) $j=2$; $j=3$; $j=4$; $j=5$ so $j=1$. Go to 3.
(3) c_{41} is minimum. $x_{41}=75=b_1$.
(4) $a_4=925$; $b_1=0$.
(5) $SC=SC-\{1\}=\emptyset$; $j=2$; $(4,1)\epsilon B$; go to 1.
(1) Go to 7.
(7) $x_{1,5}=5$; $x_{4,5}=925$; $(1,5)$, $(4,5)\epsilon B$. STOP.

The circled c_{ij} in Figure 3 incidcates the cell $(i,j)\epsilon B$. x_{ij} is written in the northeast corner of cell (i,j). It can be verified that this basis structure is primal feasible.

The following theorem has been proved. [6, 9, 10].

THEOREM 1: The simplex algorithm for a nondegenerate generalized transportation problem involves at each pivot step a primal feasible solution $X=\{x_{ij}\}$ and a basis structure (B, LB, UB), such that

(a) $(i,j)\epsilon B$ then $0<x_{ij}<U_{ij}$.
(b) $(i,j)\notin B$ then either
(i) $x_{ij}=0$ and $(i,j)\epsilon LB$ or
(ii) $x_{ij}=U_{ij}$ and $(i,j)\epsilon UB$.

Define a maximal connected subset of a basis as a *component*. It has been proved that the basis of any basic solution to a generalized transportation problem consists of mutually disconnected components where each component is a one-tree [6, 10]. Thus the basis is a one-forest.

From the algorithm for starting solution, it is easy to see that the starting solution is a one-forest consisting of one-trees with loops. Since choice of a_{m+1} is arbitrary, $x_{m+1,n+1}$ will *absorb* any possible imbalances either in the rows or columns.

Before proceeding further, let us present the graph associated with a *one-forest* corresponding to a basis B.

For a general reference to graph theory see C. Berge [7] or F. Harary [18]. For a discussion of the three-index labelling method, definitions, terminologies and algorithms for the ordinary transportation problem, see V. Srinivasan and G. L. Thompson [23] and S. Glicksman, L. Johnson, and L. Eselson [14].

The graph $G=(V, E)$ of a basic solution consists of the nodes $V=I\cup J'$ and the edges $E=B$, where B is the set of basis cells. An edge $(i, n+1)$ is a loop connecting to itself. It was seen from definition 7, that the graph $G=(V, E)=(I\cup J', B)=\bigcup_{h=1}^{k} G(T_h)$, is a *one-forest*, where the subgraph $G(T_h)=(V_h, E_h)=\{I_h\cup J'_h, B_h\}$. The graph $G(T_h)$ has certain properties. Each subgraph $G(T_h)$ has either a cycle C_h or a loop (slack cell), but not both. It has two classes of nodes I_h and J'_h and every path in the subgraph T_h makes alternate use of one, then the other kind of node.

DEFINITION 11: A *rooted-one-tree* $G(T_h)=\{I_h\cup J'_h, B_h\}$ is a one-tree with a *distance function* $d_h(v)$ for $v\epsilon V_h$ satisfying conditions (a) to (d) given below:
Choose an edge (r, s) as follows:

(i) if there is a loop $C_h=\{(r, n+1)\}$ then let $(r, s)=(r, n+1)$ or
(ii) if there is a cycle C_h, let (r, s) be any edge (basic cell) in C_h such that $(r, s)\epsilon S_{\alpha h}$ (see Definition 2); then

(a) $G_h = (I_h \cup J_h', B_h - \{r, s\})$ is a *tree.*

(b) r is the root of G_h.

(c) $d_h(v) = 0$ for all $v \in S(C_h)$.

(d) if u is any node in $G_h - C_h$ then $d_h(u)$ is defined as $d_h(u) = \min_{v \in S(C_h)} l(v, u)$;

In $G_h \in G(T_h)$ there is a unique path from the root r to any other distinct node $v \in G_h$, since G_h is a tree. Associated with a *rooted-one-tree* $G(T_h)$, the following *predecessor function* $p_h(u)$ for any node $u \in I_h \cup J_h'$ is defined.

DEFINITION 12: The *predecessor function* $p_h(u)$ for every node $u \in I_h \cup J_h'$ satisfies the following:

(a) $p_h(r) = \phi$ where r is the root of G_h

(b) $p_h(v) = u$ if $(u, v) \in E_h$ and r connected to u in $B_h - \{(r, s)\} - \{(u, v)\}$.

The following algorithm determines the distance and predecessor function in the tree $G_h = (I_h \cup J_h', B_h - \{(r, s)\}$ for the *starting** solution.

ALGORITHM A2: Algorithm for finding root, distance, and predecessor functions for the *initial** basis produced by algorithm A1.

(0) Set $r=1$; $i=0$.

(1) If $r > m+1$, STOP. Else go to (2).

(2) Check to see if $(r, n+1) \in B$. If yes go to (3). If no, $r=r+1$. Go to (1).

(3) Set $i=i+1$. Let r be the root of T_i; $p_i(r) = \phi$; $d_i(r) = 0$; $M_r = \{r\}$; $M_c = \phi$. $S_r = S_c = \phi$.

(4) For each $u \in M_r$

(a) find the set $D_c(u) = \{v | v \in J' - S_c$ and $(u, v) \in B_h\}$

(b) for each $v \in D_c(u)$ let $d_i(v) = d_i(u) + 1$ and $p_i(v) = u$.

(5) Replace S_r by $S_r \cup M_r$ and M_c by $\cup_{u \in M_r} D_c(u)$.

(6) If $M_c = \phi$, set $r=r+1$ and go to (1). Else go to (7).

(7) For each $v \in M_c$

(a) find the set $D_r(v) = \{u | u \in I - S_r$ and $(u, v) \in B_h\}$

(b) for each $u \in D_r(v)$ let $d_i(u) = d_i(v) + 1$ and $p_i(u) = v$.

(8) Replace S_c by $S_c \cup M_c$ and M_r by $\cup_{v \in M_c} D_r(v)$.

(9) IF $M_r = \phi$, set $r=r+1$ and go to (1). Else go to (4).

EXAMPLE (continued): Let us now consider how this algorithm A2 is used for the starting solution given as Figure 3 to obtain the root, distance and predecessor function for the initial basis:

Steps of A2	*Result due to this step of A2.*
(0)	$r=1$; $i=0$; $n+1=5$.
(2)	$(1, 5) \in B$.

*This algorithm A2 is only for the initial basis. For any general basis an algorithm similar to A2 (but long) is given in [4]. Further, later on [1] we will define $p_h(r) = s$ and $\neq \phi$ where (r, s) is given in Definition 11.

(3) $i = 1$; row 1 is the root of one tree T_1; p_1(row 1) $= \emptyset$; d_1 (row 1) $= 0$; $M_r = \{1\}$; $M_c = 0$ $S_r = S_c = \emptyset$.

(4) (a) $D_c(1) =$ (columns 3 and 4).

(b) d_1 (col. 3) $= 1$; d_1 (col. 4) $= 1$; p_1 (col. 3) $= p_1$ (col. 4) $=$ row 1.

(5) $S_r = \{$row 1$\}$; $M_c =$ (col. 3, col. 4).

(7) D_r(col. 3) $= D_r$(col. 4) $= \emptyset$.

(8) $S_c =$ (col. 3, col. 4); $M_r = \emptyset$.

(9) $r = 2$.

(1) 2, 3, 4 are not greater than 4.

(2) $(4, 5) \in B$.

(3) $i = 2$; Row 4 is the root of one tree T_2; p_2(row 4) $= \emptyset$; d_2(row 4) $= 0$; $M_r = \{r\}$; $M_c = \emptyset$. $S_r = S_c = \emptyset$.

(4) (a) D_c(row 4) $=$ (col. 1).

(b) d_2(col. 1) $= 1$; p_2(col. 1) $=$ row 4.

(5) $S_r = \{$row 4$\}$; $M_c =$ (col. 1).

(7) (a) D_r(col. 1) $=$ (row 2, row 3)

(b) d_2(row 2) $= d_2$(row 3) $= 2$; p_2(row 2) $= p_2$(row 3) $=$ col. 1.

(8) $S_c =$ (col. 1) and $M_r =$ (row 2, row 3).

(4) (a) D_c(row 2) $=$ col. 2; D_c(row 3) $= \emptyset$.

(b) d_2(col. 2) $= 3$ and p_2(col. 2) $=$ row 2.

(5) $S_r =$ (rows 2, 3, 4) and $M_c =$ col. 2.

(7) D_r(col. 2) $= \emptyset$.

(8) $S_c =$ (col. 2); $M_r = \emptyset$.

(9) $M_r = \emptyset$; $r = 5$.

(1) $5 > 4$: STOP.

The starting solution consists of a basis whose graph is a one-forest consisting of one-trees each having a loop. Since a loop has just one cell $(r, n + 1)$, the specification of the root of the 1-tree is obvious, viz r. However, it will be seen that when a new cell enters the basis a *two-tree* is created which may contain a cycle C_h of $2l$ $(l \geq 2)$ cells. It is necessary to identify the two sets $S_{\alpha h}, S_{\beta h}$ and determine its circulation factor for finding the new basis. The following algorithm provides this.

ALGORITHM A3: Algorithm for determining the sets $S_{\alpha h}$, $S_{\beta h}$ associated with a cycle C_h and calculation of ρ_h.

(0) Choose any edge $(r, s) \in C_h$. Store s; we now construct a tree with r being the *root*. Set $A_\alpha = e_{rs}$; $A_\beta = 1$; $S_{\alpha h} = \{(r, s)\}$; $S_{\beta h} = \emptyset$ and $v = s$.

(1) Let $u = p(v)$; if $u \neq r$ go to 2 and if $u = r$ go to 4.

(2) $S_{\beta h} = S_{\beta h} \cup \{(u, v)\}$; $A_\beta = A_\beta e_{uv}$.

(3) Let $v = p(u)$; $S_{\alpha h} = S_{\alpha h} \cup \{(u, v)\}$; $A_\alpha = A_\alpha e_{uv}$ and go to 1.

(4) $A_\beta = A_\beta e_{rv}$. $S_{\beta h} = S_{\beta h} \cup \{(r, v)\}$. If $A_\alpha = A_\beta$ go to 6. If $A_\alpha < A_\beta$ interchange $S_{\alpha h}$ and $S_{\beta h}$; also A_α and A_β. Set $s = v$ in the stored edge (r, s). Go to 5.

(5) $\rho_h = A_\alpha/(A_\alpha - A_\beta)$. STOP.

(6) This is a symmetric closed loop which can be eliminated by an ordinary pivot step, thus changing the two-tree into a one-tree, see Remark 4.

u_i \ v_j	31/5	28/5	3	4		a_i
0	3 | 15 | 3 | 15	2 | 6 | N	2 | 35 | (3) | 40	5 | 60 | (4) | 80	1 | 5 | (0) | N	300
-1/5	1 | 135 | (6) | 150	3 | 30 | (5) | 75	5 | 8 | 25	10 | 15 | N	1 | 0 | N	225
-13/10	4 | 20 | (1) | 60	2 | 30 | (3) | N	2 | 10 | 30	1 | 6 | 50	1 | 0 | N	140
0	1 | 100 | N	1 | 100 | N	1 | 100 | N	1 | 100 | N	1 | 1000 | (0) | N	1000
b_j	170	60	35	60	-	

FIGURE 4.

EXAMPLE (continued): Let us now apply algorithm (A3) for the cycle $\{(2, 1), (2, 2), (3, 1), (3, 2)\}$ given in Figure 4 to find S_α, S_β and ρ. (Subscript h is dropped for convenience here.)

Let (2, 1) be the edge chosen. Column 1 is stored; Row 2 is the root; $A_\alpha = e_{21} = 1$; $A_\beta \equiv 1$. $S_\alpha = \{(2, 1)\}$; $S_\beta = \emptyset$; col. v = col. 1.

Step *Result due to this step of A3.*

(1) row 3 = p(col. 1); row 3 $\neq$ row 2.

(2) $S_\beta = (3, 1)$; $A_\beta = 1 \times e_{31} = 4$.

(3) col. 2 = p(row 3); $S_\alpha = \{(2, 1), (3, 2)\}$; $A_\alpha = A_\alpha \times e_{32} = 2$.

(1) row 2 = p(col. 2); row 2 = row 2.

(4) $A_\beta = A_\beta \times e_{22} = 4 \times 3 = 12$; $S_\beta = \{(3, 1), (2, 2)\}$. Since $A_\alpha = 2 < A_\beta = 12$ interchange S_α and S_β and s = col. 2 and thus (2, 2) is the edge removed. Thus $S_\alpha = \{(3, 1), (2, 2)\}$ $S_\beta = \{(2, 1), (3, 2)\}$ and $\rho = A_\alpha/(A_\alpha - A_\beta) = 12/10 = 6/5$. STOP.

REMARK 7: Let $T_1, T_2, \ldots, T_h, \ldots T_k$ be the one-trees of a basis B and let $C_1, C_2, \ldots, C_h, \ldots, C_k$ be their respective cycles or loops such that $C_h \subseteq T_h$. Consider a specific one-tree T_h, and the *rooted one-tree* $G(T_h) = (I_h \cup J_h', B_h)$ with $(r, s) \in S_{\alpha h}$ being the edge removed from B_h to form the tree G_h (Definition 11). The distance function $d_h(v)$ and predecessor function $p_h(v)$ for $v \in I_h \cup J_h'$ are known from algorithm A2. Let $|I_h|$ and $|J_h'|$ represent the number of elements in I_h and J_h', respectively. It follows from Definition 7 that $I = \bigcup_{h=1}^{k} I_h$ and $J' = \bigcup_{h=1}^{k} J_h'$ with $|I| = m + 1$ and $|J'| = n$. Moreover $I_h \cap I_t = J_h' \cap J_t' = \emptyset$ for $h \neq t$.

Let (e, f) be a cell not in B. Then (e, f) is either in LB or in UB. The basic pivot step of the simplex method, or of the operator theory to be developed here, involves inserting (e, f) into B. When (e, f) is added to B two cases arise:

(a) $e \in I_h$ and $f \in J_t'$ where $h \neq t$;

(b) $e \in I_h$ and $f \in J_t'$ with $h = t$.

Since $e \in I_h$ in either case $p_h(e) = j$ is a column which is connected in T_h to C_h. Let D_h consist of the cell (e, f) together with the unique path in T_h that connects j to u, where $u \in S(C_h)$ and $l(u, j)$ is minimum. (If $j = u$, then $D_h = \{(e, f)\}$). We call D_h the *row path*.

Similarly, since $f \epsilon J_t$ in either case, $p_t(f) = i$ is a row which is connected in T_t to C_t. Let D_t consist of the cell (e, f) together with the unique path in T_t that connects i to v where $v \epsilon S(C_t)$ and $l(i, v)$ is minimum. (If $i = v$ then $D_t = \{(e, f)\}$). We call D_t the *column path*.

Hence, due to the introduction of the cell $(e, f) \notin B$, into the basis, a set of cells Γ can be determined, where

$$\Gamma = (C_h \cup D_h) \cup (D_t \cup C_t). \tag{10}$$

In this, $(C_h \cup D_h)$ is called the *row segment* of Γ while $(D_t \cup C_t)$ is called the *column segment* of Γ. The graph associated with the edges given by Γ and the nodes in the span of the cells of Γ, $(S(\Gamma))$ is called a *two-tree*.

It has been proved that this graph contains exactly two cycles (lemma 9 of [6]). It is possible that both the row segment and the column segment are identical.

Let us consider changes in x_{ef}. It can be shown [6, 10] that such changes can be compensated for by changing only x_{ij} for $(i, j) \epsilon \Gamma$. In the next algorithm (A4), we shall assume that $(e, f) \epsilon LB$, (the alternate case of $(e, f) \epsilon UB$ is covered in Remark 8), so that initially $x_{ef} = 0$ and we want to make it positive.

DEFINITION 13: If x_{ef} is changed, then Γ can be partitioned into two sets Γ_1 and Γ_2, such that x_{ij} must be increased by an amount μm_{ij} for $(i, j) \epsilon \Gamma_1$, and x_{ij} must be decreased by an amount μm_{ij} for $(i, j) \epsilon \Gamma_2$.

The following algorithm (A4) determines Γ_1 and Γ_2, the cell $(e_1, f_1) \epsilon \Gamma$ to leave the basis, and also finds the coefficients m_{ij} by which the x_{ij}'s must be revised for $(i, j) \epsilon \Gamma$, for the case when $(e, f) \epsilon LB$.

ALGORITHM A4: For finding m_{ur}, Γ_1, Γ_2, μ, x_{ur} for $(u, v) \epsilon \Gamma$ and the *outgoing cell* (e_1, f_1) when cell $(e, f) \epsilon LB$ is added to B. (If $(e, f) \epsilon UB$ make the changes indicated in Remark 8.)

(0) Let $m_{ur} = 0$ for $(u, v) \epsilon \Gamma$. Let $(e, f) \epsilon \Gamma_1$, $x = 1$, and $m_{ef} = +1$. Let $(u, v) = (e, f)$. Set $F1 = 0$ indicating we are evaluating a row segment. Let the span of the cycle $S(C) = SR \cup SC$.

(1) If $u \epsilon SR$, i.e., $d(u) = 0$ go to (5), else go to (2).

(2) Let $j = p(u)$. Let $x = -x \dfrac{e_{ur}}{e_{uj}}$, $m_{uj} = m_{uj} + x$. If $m_{uh} \geq 0$, let $(u, j) \epsilon \Gamma_1$, else $(u, j) \epsilon \Gamma_2$. Let $v = j$.

(3) If $v \epsilon SC$, i.e., $d(v) = 0$, go to (8); else go to (4).

(4) Let $i = p(v)$. Let $x = -x$, $m_{iv} = m_{iv} + x$; if $m_{iv} \geq 0$ let $(i, v) \epsilon \Gamma_1$; else $(i, v) \epsilon \Gamma_2$. Let $u = i$. Go to (1).

(5) Let $j = p(u)$. If $j = n + 1$, let $m_{u, n+1} = m_{u, n+1} - e_{uv} m_{uv}$. If $F1 = 1$ go to (11). If $F1 = 0$ let $F1 = 1$, $(u, v) = (e, f)$, $x = 1$ and go to (3). If $j \neq n + 1$, (if $j = \emptyset$, $j = s$) let $x = -x\rho \dfrac{e_{ur}}{e_{uj}}$, $m_{uj} = m_{uj} + x$. If $m_{uj} \geq 0$, let $(u, j) \epsilon \Gamma_1$; else $(u, v) \epsilon \Gamma_2$. Let $SC = SC - \{j\}$ and $v = j$.

(6) If $SR = \emptyset$ and $F1 = 0$, let $(u, v) = (e, f)$, $x = 1$, $F1 = 1$, go to (3); if $SR = \emptyset$ and $F1 = 1$ go to (11). Else let $i = p(v)$; let $x = -x$, $m_{ir} = m_{ir} + x$. If $m_{iv} \geq 0$ let $(i, v) \epsilon \Gamma_1$, else $(i, v) \epsilon \Gamma_2$. Let $SR = SR - \{i\}$ and $u = i$.

(7) If $SC = \emptyset$ and $F1 = 0$, let $F1 = 1$, let $(u, v) = (e, f)$, $x = 1$, go to (3); if $SC = \emptyset$ and $F1 = 1$ go to (11). Else let $j = p(u)$: if $j = \emptyset$, let $j = s$; let $x = -x \dfrac{e_{uv}}{e_{uj}}$, $m_{uj} = m_{uj} + x$. Let $SC = SC - \{v\}$ and $v = j$. Go to (6).

(8) Let $i=p(v)$. Let $x=-x\rho$ and $m_{iv}=m_{iv}+x$. If $m_{iv}\geq 0$ let $(i, v)\epsilon\Gamma_1$; else $(i, v)\epsilon\Gamma_2$. Let $SR=SR-\{i\}$. Let $u=i$.

(9) If $SR=\emptyset$ and $F1=0$, let $(u, v)=(e, f)$; $x=1$, and $F1=1$, go to (3); if $SR=\emptyset$ and $F1=1$ go to (11). Else let $j=p(u)$; if $j=\emptyset$, let $(u, j)\epsilon\Gamma_1$; else let $(u, j)\epsilon\Gamma_2$. Let $SC=SC-\{j\}$ and $v=j$.

(10) Let $SR=\emptyset$ and $F1=0$, let $(u, c)=(e, f)$, $x=1$ and $F1=1$, go to (3); if $SR=\emptyset$ and $F1=1$, go to (11). Else let $i=p(v)$; let $x=-x$ and $m_{iv}=m_{iv}+x$. If $m_{iv}\geq\emptyset$, let $(i, v)\epsilon\Gamma_1$; else let $(i, v)\epsilon\Gamma_2$. Let $SR=SR-\{i\}$. Let $u=i$. Go to (9).

(11) Find μ from

$$\mu=\min\begin{cases}(U_{ij}-x_{ij})/m_{ij} & \text{for } (i, j)\epsilon\Gamma_1\\ x_{ij}/(-m_{ij}) & \text{for } (i, j)\epsilon\Gamma_2.\end{cases}$$

The cell on which this minimum is taken is the *leaving cell* (e_1, f_1). If $(e_1, f_1)\epsilon\Gamma_1$, $(e_1, f_1)\epsilon UB$, and if $(e_1, f_1)\epsilon\Gamma_2$, $(e_1, f_1)\epsilon LB$. For all $(i, j)\epsilon\Gamma$, let

(12) $$x_{ij}=x_{ij}+\mu m_{ij}.$$

REMARK 8: The algorithm A4, given above provides the basic pivot step of the generalized transportation problem under the assumption that (e, f) was in LB. If (e, f) was in UB then, in the starting step (0) of algorithm A4 make the following changes: "Let $(e, f)\epsilon\Gamma_2$ and $x=-1$" and replace "$x=1$" by "$x=-1$" in steps (5)–(7), (9), (10), and continue with the rest of the algorithm.

Due to this basic pivot step, the basis structure is changed, since $(e, f)\notin B$ is now in B and $(e_1, f_1)\epsilon B$ is now not contained in B. However, it was shown [6, 10] that the resultant basis is also another one-forest. This fact is given as a theorem, the proof of which is given in [6].

THEOREM 2: The *pivot step* (Algorithm A4) of the generalized transportation problem preserves the one-forest property of the basis B.

EXAMPLE (continued): Consider Figure 3 for which when Algorithm A2 was applied, the distance and predecessor function were obtained. Assume that the cell (3, 2) is to enter the basis. Since both row 3, and column 2 are contained in the one-tree consisting of cells $\{(4, 5), (4, 1), (3, 1), (2, 1), (2, 2)\}$ we will apply algorithm A4 to the set Γ, which is a two-tree, obtained by the introduction of cell (3, 2) into the basis. Figure 5 gives the graph of Γ and the distance function. The following gives the steps of algorithm A4 and the results due to execution of the corresponding step.

Step No. *Result due to this step of A4.*

(0) $(u, v)=(3, 2)$; $m_{32}=1$; $F1=0$; $(3, 2)\epsilon\Gamma_1$; $SR=\{4\}$; $SC=\{5\}$.

(1) $d(R_3)=2\neq 0$. Go to (2).

(2) $C_1=p(R_3)$; $m_{31}=0-1\times(2/4)=-1/2$; $(3, 1)\epsilon\Gamma_2$.

(3) $d(C_1)=1\neq 0$. Go to (4).

(4) $R_4=p(C_1)$; $m_{41}=0-(-1/2)=1/2$; $(4, 1)\epsilon\Gamma_1$. Go to (1).

(1) $R_4\epsilon SR$. Go to (5).

(5) $C_5=p(R_4)$; (4, 1) is the terminal edge and (4, 5) the entry edge of row segment; $m_{45}=0-1/2=-1/2$ and $(4, 5)\epsilon\Gamma_2$. $F1=1$, $(u, v)=(3, 2)$. Go to (3).

(3) $d(C_2)=3\neq 0$. Go to (4).

(4) $R_2=p(C_2)$; $m_{22}=0-1=-1$; $(2, 2)\epsilon\Gamma_2$. Go to (1).

(1) $d(R_2)=2\neq 0$. Go to (2).

(2) $C_1=p(R_2)$; $m_{21}=0-\dfrac{(-1)(3)}{1}=3$; $(2, 1)\epsilon\Gamma_1$.

(3) $d(C_1)=1\neq 0$. Go to (4).

(4) $R_4=p(C_1)$; $m_{41}=1/2-3=-5/2$: $(4, 1)\epsilon\Gamma_2$ (note $(4, 1)$ is updated from Γ_1 to Γ_2). Go to (1).

(1) $R_4\epsilon S((4, 5))$. Go to (5).

(5) $C_5=p(R_4)$:$(4, 1)$ is terminal edge and $(4, 5)$ entry edge for column segment also; $m_{45}=-1/2-(-5/2)=2$, $(4, 5)\epsilon\Gamma_1$. Go to (11). (note that $(4, 5)$ is updated from Γ_2 to Γ_1.)

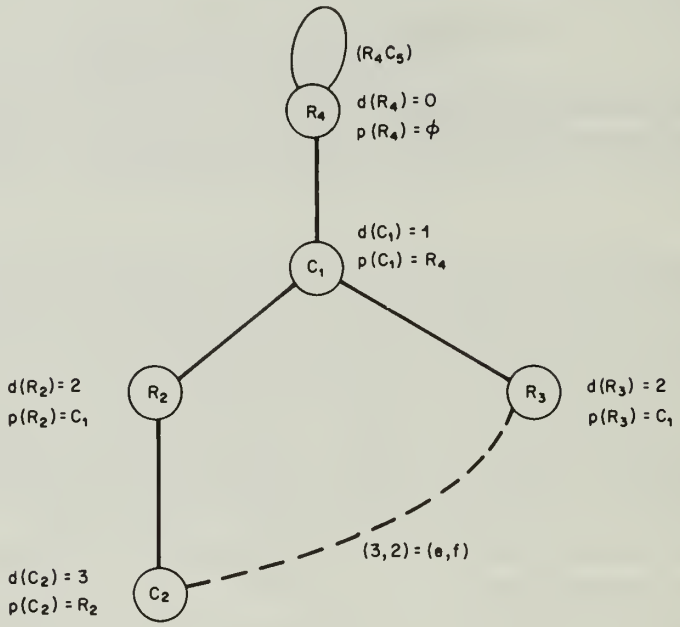


FIGURE 5.

(11) $$\mu=\min\begin{cases}\infty, & \dfrac{150-45}{3}, & \infty\\ \dfrac{60}{1}, & \dfrac{35}{(1/2)}, & \dfrac{75}{(5/2)}\end{cases}$$

$(e, f)=(4, 1)$ and $\mu=30$.

After making indicated changes we have:

$$\left.\begin{aligned}x_{32}&=30\\ x_{21}&=135\\ x_{45}&=1000\end{aligned}\right\}\epsilon\Gamma_1 \qquad \left.\begin{aligned}x_{22}&=30\\ x_{31}&=20\\ x_{41}&=0\rightarrow LB\end{aligned}\right\}\epsilon\Gamma_2.$$

Figure 4 gives the new tableau when these new x_{ij}'s given above for $(i, j)\epsilon\Gamma$ replace the old x_{ij}'s given in Figure 3. The current basis B, in Figure 4 consists of cells with c_{ij} circled, viz., $\{(1, 3), (1, 4),$

(1, 5), (2, 1), (2, 2), (3, 1), (3, 2), (4, 5)}. The cell {(1, 1)} is in its upper bound and hence is contained in *UB*. The rest of the cells {(1, 2), (2, 3), (2, 4), (2, 5), (3, 3), (3, 4), (3, 5), (4, 1), (4, 2), (4, 3), (4, 4)} are at their lower bound. This set of cells at their lower bound constitute *LB*. There are $m+n+1=8$ cells in the basis which always includes the absorbing cell (4, 5). This basis form a *one-forest* consisting of three mutually disconnected *one-trees*. The first one-tree T_1 consists of cells {(1, 3), (1, 4), (1, 5)} with the slack element {(1, 5)} $=C_1$ (loop). The cells (1, 3), (1, 4) are the *branches* of T_1. Row 1 is the root of T_1. The second *one-tree* T_2 consists of cells {(2, 1), (2, 2), (3, 1), (3, 2)} which coincides with its cycle C_2; while the third one-tree T_3 consists of only the absorbing cell (4, 5). It is easy to see that every cell $(i, j) \epsilon B$ is an element in only one of the one-trees and there is at least one path between any two lines contained in the span of a specified one-tree.

3. THE DUAL PROBLEM

Let us now define the dual problem to P. Let u_i for $i \epsilon I$, v_j for $j \epsilon J'$ and w_{ij} for $i \epsilon I$, $j \epsilon J'$ be dual variables associated with the row constraints (6), column constraints (7) and upper bound constraints (8), respectively. Then the dual problem (D) is:

$$\text{(13)} \qquad \text{Maximize} \quad \sum_{i \epsilon I} a_i u_i + \sum_{j \epsilon J'} b_j v_j - \sum_{i \epsilon I} \sum_{j \epsilon J'} U_{ij} w_{ij} = F.$$

Subject to the following constraints:

$$\text{(14)} \qquad e_{ij} u_i + v_j - w_{ij} \leqslant c_{ij} \qquad i \epsilon I;\ j \epsilon J'$$

$$\text{(15)} \qquad w_{ij} \geqslant 0 \qquad i \epsilon I;\ j \epsilon J'$$

$$\text{(16)} \qquad u_i \leqslant 0 \qquad i \epsilon I.$$

Equation (16) can be derived as follows: since $U_{i,n+1}=M$ for $j=n+1$, (14) becomes $e_{i,n+1}u_i \leqslant c_{i,n+1}$; but $c_{i,n+1}=0$, $e_{i,n+1}=1$, so (16) follows.

Given a basis B, one can determine a unique set of solutions for u_i and v_j to the equations

$$\text{(17)} \qquad e_{ij} u_i + v_j = c_{ij} \text{ for } (i, j) \epsilon B$$

so that $d_{ij}=e_{ij}u_i+v_j$ is unique for all (i, j); $i \epsilon I$; $j \epsilon J'$. Because of (16) $d_{i,n+1}=u_i \leqslant 0$.

REMARK 9: Unlike the ordinary transportation problem, u_i and v_j are unique, because the dual variables are uniquely determined by the one-tree in whose span they lie. Moreover in the optimal solution, if line i contains a basis cell in the slack column then $u_i=0$. Since the absorbing cell is always in the basis [10], $u_{m+1}=0$. In addition all other u_i must be nonpositive at the optimal solution, due to (16). Further, if c_{ij} and e_{ij} are all positive, then because of (17) the optimal v_j's will be nonnegative.

REMARK 10: Since the one-trees are mutually disconnected in the basis graph $G(B)$, u_i and v_j associated with the span of $S(T_h)$ of T_h depend only on the basis cells contained in T_h. Thus they are independent of the dual variables associated with any other one-tree, T_g where $g \neq h$. Hence, in the basic pivot step, if some one-trees of the basis of preceeding iteration are not changed, the associated u_i and v_j are also not changed.

The following algorithm provides a method of evaluation u_i, $i \epsilon I$ and v_j, $j \epsilon J'$ for the initial solution given by A2.

ALGORITHM A5: For finding the dual variable u_i, v_j associated with *initial** basic solution.

(0) Let initial basis B consist of k trees T_h with predecessor and distance functions p_h, d_h for $h=1, \ldots, k$. Let $h=1$.

(1) Let r be the root of T_h, i.e., $d_h(r)=0$. Let $u_r=0$; $t=1$.

(2) Find $SC_t=\{j \mid j\epsilon J_h'$ and $d_h(j)=t\}$.
If $SC_t=\emptyset$, go to 6.

(3) For each $j\epsilon SC_t$ let $i=p_h(j)$ and let $v_j=c_{ij}-e_{ij}u_i$. Let $t=t+1$.

(4) Find $SR_t=\{i \mid i\epsilon I_h$ and $d_h(i)=t\}$. If $SR_t=\emptyset$, go to 6.

(5) For each $i\epsilon SR_t$, let $j=p_h(i)$ and let $u_i=(c_{ij}-v_j)/e_{ij}$. Let $t=t+1$. Go to 2.

(6) If $h=k$, STOP. Else let $h=h+1$ and go to 1.

EXAMPLE (continued): Dual variables u_i and v_j for the initial basis given as Figure 3 are obtained utilizing the distance and predecessor given by A2. These are shown in Figure 3 as row and column headings.

Steps of A5 — *Result due to this step of A5.*

(0) Basis B consists of 2 trees T_1, T_2 with p_h and d_h; $h=1$; $k=2$.

(1) Row 1 is the root of T_1 and $d_1(r_1)=0$; $u_1=0$; $t=1$.

(2) $SC_1=\{C_3, C_4\}$ as $d_1(C_3)=d_1(C_4)=1$.

(3) $p_1(C_3)=p_1(C_4)=r_1$; $v_3=3=0$; $v_4=4=0$; $t=2$.

(4) $SR_t=\emptyset$. Go to (6).

(6) $h=2$. Go to (1).

(1) Row 4 is the root of T_2 as $d_2(r_4)=0$; $u_1=0$; $t=1$.

(2) $SC_1=\{$col. 1$\}$ since $d_2(C_1)=1$.

(3) $p_1(C_1)=r_4$; $v_1=100$; $t=2$.

(4) $SR_2=\{$rows 2, 3$\}$ since $d_2(r_2)=d_2(r_3)=2$.

(5) $j=$ col. 1. $u_2=(6-100)1=-94$; $u_3=(1-100)/4=-99/4$. $t=3$. Go to (2).

(2) $SC_3=\{C_2\}$ since $d_2(C_3)=3$.

(3) row $2=p_2(C_2)$, $v_2=5-3\times(-94)=287$; $t=4$.

(4) $SC_3=\emptyset$. Go to (6).

(6) $h=2=k$. STOP.

Thus $[u_1, u_2, u_3, u_4]=[9, -94, -99/4, 0]$ and $[v_1, v_2, v_3, v_4]=[100, 287, 3, 4]$ for the initial distribution. These are given as row and column headings of Figure 3, respectively.

Both the algorithms A2 and A5 discuss the determination of the primal and dual solutions. In [4] we will discuss algorithms for changing the basis from a given one-forest to the new one required at a pivot step. These are based on similar algorithms for the ordinary transportation problem [23].

The matrix $D=\{d_{ij}\}$ is called the *dual matrix*. It is known that [9, 20] a primal basic feasible solution is *optimal* if its dual solutions satisfy

(18) $d_{ij}=e_{ij}u_i+v_j \leq c_{ij}$ for $(i, j)\epsilon LB$ and

(19) $d_{ij}=e_{ij}u_i+v_j \geq c_{ij}$ for $(i, j)\epsilon UB$.

*This algorithm is only for the initial basis. However a similar algorithm (though long) for any general basis is given in [4].

If equality occurs in either of these two equations then we say the solution is *dual degenerate*. The usual techniques [20] can be used to prevent dual degeneracy.

A basis B, more precisely a basis structure (B, LB, UB) is said to be *dual feasible* if u_i and v_j determined from (17) satisfy (18) and (19). If we define

$$w_{ij} = \max(0, e_{ij}u_i + v_j - c_{ij}) \tag{20}$$

which means $w_{ij}=0$ for $(i,j) \notin UB$) for $i \in I$ and $j \in J'$ it may be verified that (17)–(19) imply (14) and (15).

From the duality theorem of linear programming [9, 20], a basic solution is optimal if it is both primal and dual feasible. Furthermore for such a solution,

$$Z = \sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij} = \sum_{i\in I} a_i u_i + \sum_{j\in J} b_j v_j - \sum_{i\in I'}\sum_{j\in J'} U_{ij}w_{ij} = F. \tag{21}$$

EXAMPLE (continued): Consider Figure 4 where the cells with c_{ij} circled give the basis. In the northeast corner of a cell (i, j) is given x_{ij} and it could be verified that this solution is basic and primal feasible. The dual variables u_i and v_j appearing at the left and top rims, respectively, satisfy (17). Unlike the ordinary transportation problem u_i and v_j are unique. The dual matrix $D=\{d_{ij}=e_{ij}u_i+v_j\}$ is given as Figure 6. The reader may verify that the solution is dual feasible as well, i.e., satisfies (18) and (19) with $LB=\{(1,2),(2,3),(2,4),(2,5),(3,3),(3,4),(3,5),(4,1),(4,3),(4,4)\}$ and $UB=\{(1,1)\}$. Since this tableau given as Figure 4 is both primal and dual feasible, it is *optimal*. Setting w_{ij} as given by (20) ($w_{ij}=0$ for $(i,j) \neq UB$), it could be verified the $Z=F=\$1{,}460$ as given in (21). In this optimal tableau (Figure 4) every u_i with a cell, $(i, n+1) \in B$ is zero, (e.g., the absorbing cell). Further, all other $u_i < 0$ as $c_{i,n+1}=0$. As we have both costs, c_{ij}, and the weights e_{ij} are nonnegative, due to (17) all v_j's are also nonnegative.

D: $\{d_{ij} = e_{ij}u_i + v_j\}$

d_{ij} GIVEN IN THE CENTER AND e_{ij} GIVEN IN NORTHWEST CORNER OF A CELL (i,j) $i \in I$; $j \in J'$:

u_i \ v_j →	31/5	28/5	3	4
0	3 | 31/5	2 | 28/5	2 | 3	3 | 4
-1/5	1 | 6	3 | 5	5 | 2	10 | 2
-13/10	4 | 1	2 | 3	2 | 2/5	1 | 27/10
0	1 | 31/5	1 | 28/5	1 | 3	1 | 4

FIGURE 6.

4. OPERATORS AND ASSOCIATED SOLUTIONS

Let us now consider operators that transform the optimum solutions when the data of the problem are changed as a (linear) function of single parameter.

DEFINITION 14: An *operator* $\delta\mathbf{T}(P)$ transforms the optimum solution of a problem P into that for problem $P^{\mathbf{T}}$ with data:

$$(22)\qquad \begin{cases} a_i^{\mathbf{T}} = a_i + \delta\alpha_i & \text{for } i\epsilon I' \\ b_j^{\mathbf{T}} = b_j + \delta\beta_j & \text{for } j\epsilon J' \\ c_{ij}^{\mathbf{T}} = c_{ij} + \delta\gamma_{ij} & \text{for } i\epsilon I';\ j\epsilon J' \\ e_{ij}^{\mathbf{T}} = e_{ij} + \delta\epsilon_{ij} & \text{for } i\epsilon I';\ j\epsilon J' \\ U_{ij}^{\mathbf{T}} = U_{ij} + \delta\nu_{ij} & \text{for } i\epsilon I';\ j\epsilon J', \end{cases}$$

where a_i, b_j, c_{ij}, e_{ij}, and U_{ij} are the data of the problem P, δ is nonnegative and α_i, β_j, γ_{ij}, ϵ_{ij} and ν_{ij} are given numbers, such that the transformed data satisfy assumptions (A1) and (A2). We denote this transformation as

$$(23)\qquad \delta^{\mathbf{T}}(P, B, LB, UB, X, D, Z) = (P^{\mathbf{T}}, B^{\mathbf{T}}, LB^{\mathbf{T}}, UB^{\mathbf{T}}, X^{\mathbf{T}}, D^{\mathbf{T}}, Z^{\mathbf{T}}),$$

where B, LB, UB, X, D, Z correspond to optimum solution of problem $P^{\mathbf{T}}$and $B^{\mathbf{T}}$, $LB^{\mathbf{T}}$, $UB^{\mathbf{T}}$, $X^{\mathbf{T}}$, $D^{\mathbf{T}}$, $Z^{\mathbf{T}}$ correspond to those for problem $P^{\mathbf{T}}$. In stating (22) we assume problem $P^{\mathbf{T}}$ is also of type P satisfying assumptions A1 and A2. (It is of course possible that $P^{\mathbf{T}}$ has no primal feasible solution.

REMARK 11: In the definition of the operator we have assumed that $\delta \geq 0$. This involves no loss of generality, since to study the effects of δ being negative, we can define another operator $\delta'\mathbf{T}(P)$ with $\alpha_i' = -\alpha_i$; $\beta_j' = -\beta_j$; $\gamma_{ij}' = -\gamma_{ij}$; $\epsilon_{ij}' = -\epsilon_{ij}$ and $\nu'_{ij} = -\nu_{ij}$ and still have $\delta' \geq 0$.

For most practical applications an operator as general as that of definition 13 is unnecessary. In [1] we consider *rim operators* $(\delta\mathbf{R}(P))$ which arise when *only* the rim conditions are changed, i.e., $\nu_{ij} = \epsilon_{ij} = \nu_{ij} = 0$ for all $i\epsilon I'$; $J\epsilon J'$. *Cost operators* $(\delta(C(P)))$ are also taken up in [1] in Section 3, wherein the data correspond to cost entries alone i.e., $\alpha_i = \beta_j = \epsilon_{ij} = \nu_{ij} = 0$ for $i\epsilon I'$; $j\epsilon J'$. Weight operators $\delta\mathbf{E}(P)$ that arise when only e_{ij}'s are changed (i.e., $\alpha_i = \beta_j = \gamma_{ij} = \nu_{ij} = 0$ for $i\epsilon I'$, $j\epsilon J'$) are discussed in [2].

The *bound operators* $\delta L(P)$ which result when only the U_{ij}'s change (i.e., $\alpha_i = \beta_j = \gamma_{ij} = \epsilon_{ij} = 0$ for $i\epsilon I'$, $j\epsilon J'$) are discussed in [1]. It is shown that these are equivalent to rim operators. Finally simultaneous application of combinations of operators are also given.

We call the above operators *area operators* to distinguish from *cell operators*. For a cell operator either a single cost entry c_{pq} or a single weight entry e_{pq} is changed, i.e., γ_{ij} or $\epsilon_{ij} = 0$ for all $(i, j) \neq (p, q)$, (cell cost operator or cell weight operator). Similarly when a single a_p and b_q alone are changed, i.e., $\alpha_i = 0$ for $i \neq p$ and $\beta_j = 0$ for $j \neq q$ (cell rim operator).

Surprisingly enough many of the applications of operator theory require only cell operators for which reason this is considered separately here in developing efficient algorithms. The cell operators are further classified into positive and negative operators depending on whether the data are increased or decreased.

DEFINITION 15: An operator $\delta\mathbf{T}(P)$ is said to be *basis preserving*, and denoted by light face letters $\delta T(P)$, if the transformed problem P^T has an optimum solution with $B^T = B$, $LB^T = LB$ and $UB^T = UB$ (i.e., the basis structure is preserved).

We denote by μ^T the maximum value for δ (i.e., $0 \leq \delta \leq \mu^T$) so that the operator is basis preserving. In papers [1] and [2] our discussion is limited to such operators. In [3] we show that any operator can be expressed as a product of basis preserving operators.

Table 1 summarizes the operator classification discussed above.

TABLE 1. *Operator Classification*

Symbol for Basis-Preserving operator [a]	Name	All the data for the transformed problem are the same as the original problem *except*	Notation for transformed problem [b]	Constraints on δ for basis preserving operator
$\delta T(P)$	operator	See eq. (22)	$P^T, B^T, LB^T, UB^T, X^T, D^T, Z^T$	$0 \leq \delta \leq \mu^T$
$\delta R_A(P)$	area rim operator	$a_i^A = a_i + \delta\alpha_i$ for $i \in I$ $b_j^A = b_j + \delta\beta_j$ for $j \in J'$	$P^A, B^A, LB^A, UB^A, X^A, D^A, Z^A$	$0 \leq \delta \leq \mu^A$
$\delta R_{pq}^+(P)$	(plus) cell rim operator	$a_p^+ = a_p + e_{pq}\delta$ $b_q^+ = b^q + \delta$	$P^+, B^+, LB^+, UB^+, X^+, D^+, Z^+$	$0 \leq \delta \leq \mu^+$
$\delta R_{pq}^-(P)$	(minus) cell rim operator	$a_p^- = a_p - e_{pq}\delta$ $b_q^- = b_q - \delta$	$P^-, B^-, LB^-, UB^-, X^-, D^-, Z^-$	$0 \leq \delta \leq \mu^-$
$\delta C_A(P)$	area cost operator	$c_{ij}^A = c_{ij} + \delta\gamma_{ij}$ for $i \in I$ and $j \in J$	$P^A, B^A, LB^A, UB^A, X^A, D^A, Z^A$	$0 \leq \delta \leq \mu^A$
$\delta C_{pq}^+(P)$	(plus) cell cost operator	$c_{pq}^+ = c_{pq} + \delta$	$P^+, B^+, LB^+, UB^+, X^+, D^+, Z^+$	$0 \leq \delta \leq \mu^+$
$\delta C_{pq}^-(P)$	(minus) cell cost operator	$c_{pq}^- = c_{pq} - \delta$	$P^-, B^-, LB^-, UB^-, X^-, D^-, Z^-$	$0 \leq \delta \leq \mu^-$
$\delta E_A(P)$	area weight operator	$e_{ij}^A = e_{ij} + \delta\epsilon_{ij}$ for $i \in I', j \in J'$	$P^A, B^A, LB^A, UB^A, X^A, D^A, Z^A$	$0 \leq \delta \leq \mu^A$
$\delta E_{pq}^+(P)$	(plus) cell weight operator	$e_{pq}^+ = e_{pq} + \delta$	$P^+, B^+, LB^+, UB^+, X^+, D^+, Z^+$	$0 \leq \delta \leq \mu^+$
$\delta E_{pq}^-(P)$	(minus) cell weight operator	$e_{pq}^- = e_{pq} - \delta$	$P^-, B^-, LB^-, UB^-, X^-, D^-, Z^-$	$0 \leq \delta \leq \mu^-$
$\delta L_A(P)$	area bound operator	$U_{ij}^A = U_{ij} + \delta\nu_{ij}$ for $i \in I', j \in J'$	$P^A, B^A, LB^A, UB^A, X^A, D^A, Z^A$	$0 \leq \delta \leq \mu^A$
$\delta L_{pq}^+(P)$	(plus) cell bound operator	$U_{pq}^+ = U_{pq} + \delta$	$P^+, B^+, LB^+, UB^+, X^+, D^+, Z^-$	$0 \leq \delta \leq \mu^+$
$\delta L_{pq}^-(P)$	(minus) cell	$U_{pq}^- = U_{pq} - \delta$	$P^-, B^-, LB^-, UB^-, X^-, D^-, Z^-$	$0 \leq \delta \leq \mu^-$

[a] We use bold face letters **T**, **R**, **C** to denote operators (not necessarily basis-preserving).
[b] The notation (P, B, LB, UB, X, D, Z) is used for the original problem.

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AN OPERATOR THEORY OF PARAMETRIC PROGRAMMING FOR THE GENERALIZED TRANSPORTATION PROBLEM: II RIM, COST AND BOUND OPERATORS*

V. Balachandran

Graduate School of Management
Northwestern University

and

G. L. Thompson

Graduate School of Industrial Administration
Carnegie-Mellon University

ABSTRACT

This paper investigates the effect on the optimum solution of a capacitated generalized transportation problem when certain data of the problem are continuously varied as a linear function of a single parameter. First the rim conditions, then the cost coefficients, and finally the cell upper bounds are varied parametrically and the effect on the optimal solution, the associated change in costs and the dual changes are derived. Finally the effect of simultaneous changes in both cost coefficients and rim conditions are investigated. Bound operators that effect changes in upper bounds are shown to be equivalent to rim operators. The discussion in this paper is limited to basis preserving operators for which the changes in the data are such that the optimum bases are preserved.

1. INTRODUCTION

In this paper we shall investigate the development of an "operator theory" of parametric programming for the generalized transportation problem that was introduced in section 4 of [24]. The reader is assumed to be fully familiar with Reference [24] before attempting a study of this paper.

We will consider operators which are basis preserving (Definition 15 in [24]) i.e., the original problem P and the transformed problem P^T share a common basis structure (basis B and sets LB and UB of cells at lower and upper bounds) in their (basic) optimal solutions. In section 2, we will discuss rim operators, while in sections 3 and 4, we will present cost and bound operators respectively. Basis preserving weight operators will be considered in [2] while extension of these basis preserving operators to global operators will be discussed in [3]. In this paper we shall use the same notation, bibliography and examples as in [24] and will continue from that paper the numbering of theorems, remarks, equations, algorithms, figures, etc.

Hereafter we will assume that $p_h(r) = s$ instead of ϕ where r is the root of the one-tree T_h and

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(r, s) as the edge removed in the definition of the rooted one-tree, G_h. (Vide Definition 12(a) [24].)

2. BASIS PRESERVING RIM OPERATORS

In this section let us consider rim operators that transform the optimal solutions when only the rim conditions a_i and b_j are changed, e.g.

$$\delta R^+_{pq}: a_p \rightarrow a_p + \delta e_{pq};\ b_q \rightarrow b_q + \delta;\ \text{also } \delta R^-_{pq} \text{ and } \delta R^A$$

THEOREM 3: For rim operators, the dual solutions are obtained as follows:

(a) For the operator δR^+_{pq}, the solution $D^+ = D$, $w^+_{ij} = w_{ij}$ for $i \epsilon I'$, and $J \epsilon J'$ is dual feasible with $Z^+ = Z + \delta d_{pq}$.

(b) For the operator δR^-_{pq}, the solution $D^- = D$, $w^-_{ij} = w_{ij}$ for $i \epsilon I'$ and $j \epsilon J'$ is dual feasible with $Z^- = Z - \delta d_{pq}$.

(c) For the operator δR_A, the solution $D^A = D$, $w^A_{ij} = w_{ij}$ for $i \epsilon I'$ and $j \epsilon J'$ is dual feasible with $Z^A = Z + \delta \left[\sum_{i \epsilon I'} e_{ij} \alpha_i u_i + \sum_{j \epsilon J'} \beta_j v_j \right]$.

PROOF: (a) The duals u_i, v_j and w_{ij} depend only on the cost entries (which do not change for a rim operator) and the locations of cells B, LB and UB. (Cf. (17) and (20)). Since the operator is basis preserving $B^+ = B$. This leads to the fact that $D^+ = D$ and $w^+_{ij} = w_{ij}$ for $i \epsilon I'$ and $j \epsilon J'$. By the basis preserving property $LB^+ = LB$ and $UB^+ = UB$. Thus the solution is dual feasible (i.e., satisfies (18) and (19) for problem P^+) since it holds for problem P.

At the optimum the values of primal objective function Z^+ and dual objective function F^+ are the same as that from (21)

$$Z^+ - Z = F^+ - F = \sum_{i \epsilon I} (a^+_i - a_i) u_i + \sum_{j \epsilon J'} (b^+_j - b_j) v_j - \sum_{i \epsilon I'} \sum_{j \epsilon J'} (U^+_{ij} - U_{ij}) w_{ij}.$$

By the definition of δR^+_{pq} (Table 1 [24]) $a^+_i = a_i$ for $i \neq p$ with $a^+_p = a_p + e_{pq}\delta$; $b^+_j = b_j$ for $j \neq q$ with $b^+_q = b_q + \delta$ and $U^+_{ij} = U_{ij}$ for $i \epsilon I$ and $j \epsilon J$ so that

$$Z^+ - Z = \delta u_p e_{pq} + \delta v_q = \delta(e_{pq} u_p + v_p) = \delta d_{pq}.$$

The proofs for (b) and (c) are similar.

The next three theorems give the primal solutions to the problems obtained after applying rim operators of various kinds.

THEOREM 4: (a) For the rim operator δR^+_{pq} with $(p, q) \epsilon B$, the optimal primal solution X^+ to the problem P^+ is given by

(24)
$$x^+_{ij} = x_{ij} \quad \text{for } (i, j) \epsilon [(I \times J) - \{(p, q)\}$$

$$x^+_{pq} = x_{pq} + \delta,$$

where $\times$ denotes cross products of sets. Furthermore the maximum extent to which this basis preserving

perator can be applied is given by

25) $$\mu^+ = (U_{pq} - x_{pq}).$$

(b) For the operator δR^-_{pq} with $(p, q) \in B$

26) $$x^-_{ij} = x_{ij} \quad \text{for } (i, j) \in [(I \times J)] - \{(p, q)\}$$

$$x^-_{pq} = x_{pq} - \delta$$

27) and the maximum extent μ^- is $\mu^- = x_{pq}$.

PROOF: It can be easily verified that X^+ given by (24) is a basic solution (i.e., satisfies (6) and (7) or problem P^+) and provided $\delta \leqslant \mu^+$ the solution is primal feasible (i.e., satisfies (8)). By Theorem 3(a) he solution is dual feasible also and hence optimal. The quantity μ^+ gives the maximum permissible for a basis preserving operator since for $\delta > \mu^+$, x^+_{pq} violates (8). Similar comments are applicable o (b).

REMARK 12: Consider the (cell) rim operator δR^+_{pq} with $(p, q) \notin B$. The two-tree $\Gamma = (C_h \cup D_h) \cup (D_t \cup C_t)$ as per Eq. (10) of Remark 7 can be determined and we can define Γ_1 and Γ_2 as per lgorithm A4 [24]. However, only few steps are to be modified in algorithm A4 [24] for the cell rim perator with $(p, q) \notin B$.

Table 2 gives the modifications needed in algorithm A4 after relabelling (p, q) as (e, f) to make t suitable for rim operators.

TABLE 2

Step No.	Instructions given in A4	Algorithm A4(a) for δR^+_{ef} is Algorithm A4, but modified as follows:	Algorithm A4(b) for δR^-_{ef} is Algorithm A4, but modified as follows:
(0), (5), (6), (7), (9), (10)	$x = 1$, $m_{ef} = +1$[a]	$x = -1$, $m_{ef} = -1$*	No Change ($x = 1$, $m_{ef} = 1$)
(11)	$x_{ij} = x_{ij} + \mu m_{ij}$	$x_{ij} = x_{ij} + \delta m_{ij}$	$x_{ij} = x_{ij} + \delta m_{ij}$
(Eq. (12)) in Step (11)	for $(i, j) \in \Gamma$ $\mu = \text{Min} \begin{cases} (U_{ij} - x_{ij})/m_{ij} & \text{for } (i, j) \in \Gamma_1 \\ x_{ij}/(-m_{ij}) & \text{for } (i, j) \in \Gamma_2 \end{cases}$ If $(e_1, f_1) \in \Gamma_1$, $(e_1, f_1) \in UB$ $(e_1, f_1) \in \Gamma_2$; $(e_1, f_1) \in LB$	for $(i, j) \in \Gamma - \{(e, f)\}$ $\mu^+ = \text{Min} \begin{cases} (U_{ij} - x_{ij})/m_{ij} & \text{for } (i, j) \in \Gamma_1 - \{(e, f)\} \\ x_{ij}/(-m_{ij}) & \text{for } (i, j) \in \Gamma_2 \end{cases}$ If $(e_1, f_1) \in \Gamma_2$, $(e_1, f_1) \in B$ at 0 level If $(e_1, f_1) \in \Gamma_1$, $(e_1, f_1) \in B$ at max. level	for $(i, j) \in \Gamma - \{(e, f)\}$ $\mu^- = \text{Min} \begin{cases} (U_{ij} - x_{ij})/m_{ij} & \text{for } (i, j) \in \Gamma_1 - \{(e, f)\} \\ x_{ij}/(-m_{ij}) & \text{for } (i, j) \in \Gamma_2 \end{cases}$ If $(e_1, f_1) \in \Gamma_1$, $(e_1, f_1) \in B$ at max. level If $(e_1, f_1) \in \Gamma_2$, $(e_1, f_1) \in B$ at zero level

[a] The change on $m_{ef} = +1$ is needed only in Step (0), and on $x = +1$ on all the steps listed.

These modifications lead to the following theorem.

THEOREM 5: (a) For the operator δR^{+}_{pq} with $(p, q) \notin B$,

$$x^{+}_{ij} = \begin{cases} x_{ij} + m_{ij}\delta & \text{for } i, j \in \Gamma - \{(p, q)\} \\ x_{ij} & \text{elsewhere,} \end{cases} \tag{28}$$

where Γ is a two-tree defined in (10) [24] by adding (p, q) to B, and Γ_1 and Γ_2 are the cells in Γ as given in (Remark 7) Algorithm A4 (a). The maximum extent to which this basis preserving operator can be applied is given by

$$\mu^{+} = \text{Min} \begin{cases} (U_{ij} - x_{ij})/m_{ij} & \text{for } (i, j) \in \Gamma_1 - \{(p, q)\} \\ x_{ij}/-m_{ij} & \text{for } (i, j) \in \Gamma_2. \end{cases} \tag{29}$$

(b) For the operator δR^{-}_{pq} with $(p, q) \notin B$

$$x^{-}_{ij} = \begin{cases} x_{ij} + m_{ij}\delta & \text{for } (i, j) \in \Gamma - \{(p, q)\} \\ x_{ij} & \text{elsewhere.} \end{cases} \tag{30}$$

(Note: m_{ij}'s are different from those in (28) and are given by A4 (b)).

$$\mu^{-} = \text{Min} \begin{cases} x_{ij}/-m_{ij} & \text{for } (i, j) \in \Gamma_2 \\ (U_{ij} - x_{ij})/m_{ij} & \text{for } (i, j) \in \Gamma_1 - \{(p, q)\}, \end{cases} \tag{31}$$

where Γ_1 and Γ_2 are as defined in (a), above.

PROOF: (a) We first show that X^{+} as given in (28) is a basic solution (i.e.) satisfies (6) and (7) for the problem P^{+}. From Definition 10 and Remark 7 row p has $(p, q) \in \Gamma_1$ and a cell say (p, h) belonging to Γ_2. From (28) $x^{+}_{pq} = x_{pq}$ and $x^{+}_{ph} = x_{ph} + \frac{e_{pq}}{e_{ph}}\delta$, $\left(m_{ph} = \frac{e_{pq}}{e_{ph}}\right)$. Since there are no other cells of Γ in line p, (assuming (p, q) is not the only edge in D_h, otherwise the circulation factor ρ and weights of the two cells in line p of cycle C_h balance themselves satisfying the constraint for row p).

$$\sum_{j \in J} e_{pj} x^{+}_{pj} = \sum_{j \in J} e_{pj} x_{pj} + e_{pq}\delta.$$

For other rows of $i \neq p$ from (28) and due to existence of two columns in each row one in Γ_1 and one in Γ_2 with balancing coefficients m_{ij} as found by the algorithm, it is easy to see that $\sum_{j \in J} e_{ij} x^{+}_{ij} = \sum e_{ij} x_{ij}$.

(The multipliers m_{ij} are determined by remark 7, and algorithm 4(a) as balancing coefficients so that the coefficient of δ when the two terms in each row (or column) are put together balance the rim requirements.) Since X is a basic solution to P, it satisfies (6), viz.

$$\sum_{j\in J} e_{ij}x_{ij}=a_i; \qquad \text{thus } \sum_{j\in J} e_{ij}x^+_{ij}=a_i \qquad \text{for } i\neq p \qquad \text{and } \sum_{j\in J} e_{pj}x^+_{pj}=a_p+e_{pq}\delta=a^+_p.$$

Similarly, $\sum_{i\in I} x^+_{ij}=b_j$ for $j\neq q$ and $\sum_{i\in I} x_{iq}=b_q+\delta=b^+_q$. It then easily follows that X^+ is a basic solution. Also by the choice of μ^+ in (29) the solution X^+ for $\delta\leq\mu^+$ can be verified to be primal feasible. By Theorem 3(a) the solution is dual feasible as well and hence optimal. The quantity μ^+ gives the maximum δ for basis preserving operator since for $\delta>\mu^+$, x^+_{ij} corresponding to the *leaving cell* $(i, j)=(e_1, f_1)$ (i.e., the cell at which (29) attains the minimum) can be shown to violate (8).

The proof of part (b) is similar.

To consider (basis preserving) area rim operator, we determine a *modification matrix* Y such that, when x_{ij} is replaced by $x_{ij}+\delta y_{ij}$, the conditions (6) and (7) are satisfied for the problem P^A.

DEFINITION 16: Given a basis B, the modification matrix $Y=\{y_{ij}\}$ associated with the coefficients α_i for $i\in I$ and β_j for $j\in J'$ is such that $y_{ij}=0$ for $(i, j)\notin B$, $(\alpha_{m+1}=0)$ and

$$\sum_{j\in J} e_{ij}y_{ij}=\alpha_i \qquad \text{for } i\in I, \text{ and} \tag{32}$$

$$\sum_{i\in I} y_{ij}=\beta_j \qquad \text{for } j\in J'. \tag{33}$$

REMARK 13: The determination of y_{ij} is not identical to the determination of x_{ij}, where we started with an initial solution due to algorithm A1 and proceeded toward optimal solution. Further, α_i, β_j, and y_{ij}, unlike a_i, b_j, and x_{ij}, can be negative. Though x_{ij} can be nonzero for nonbasic cells, (e.g., $x_{ij}=U_{ij}$) we require $y_{ij}=0$ for nonbasic cells. To determine the amounts y_{ij} corresponding to basic cells, we have to identify two possible cases, viz., the lines which have a unique cell and the case of lines which have more than one cell. The following algorithm determines y_{ij} corresponding to cells $(i, j)\in B$ given α_i and β_j for $i\in I, j\in J'$.

ALGORITHM A6: For determining y_{ij} corresponding to cells $(i, j)\in B$ given α_i and β_j for $i\in I$ and $j\in J'$, utilizing distance and predecessor functions for each of the one-tree T_h of the basis.

(0) Let the basis $B=\{B_1, B_2, \ldots, B_k\}$ consist of one-trees $T_1, T_2, \ldots, T_k$. Let $L=\{1, 2, \ldots, k\}$; set $S(T_h)=I_h\cup J'_h, h\in L$; and let C_h be the cycle of $T_h, h\in L$. Set $h=1$.

(1) If $h=k+1$ STOP. Let r be the root of T_h (i.e., $p_h(r)=s$). If $(r, n+1)\in B_h$, let $F1=0$ and go to (2), else let $F1=1$ and go to (5).

(2) Find a line $\in I_h\cup J'_h$ such that there is a unique cell $\in B_h$. If the cell is in column $n+1$ let $y_{i,n+1}=\alpha_i$, let $h=h+1$ and go to (1). Else let the unique cell be (i, j) and if it lies in row $i\in I_h$ go to (3), or, if $j\in J'_h$ go to (4).

(3) Let $y_{ij}=\alpha_i/e_{ij}$ and $\beta_j=\beta_j-y_{ij}$; $B_h=B_h-\{(i, j)\}$ and $I_h=I_h-\{i\}$. If $F1=0$, go to (2); if $F1=1$ go to (5).

(4) Let $y_{ij}=\beta_j$ and $\alpha_i=\alpha_i-e_{ij}y_{ij}$; $B_h=B_h-\{(i, j)\}$ and $J'_h=J'_h-\{j\}$. If $F1=0$ go to (2); if $F1=1$ go to (5).

(5) Find a line $\in I_h\cup J'_h$ such that there is a unique cell $\in B_h$. If there is no such line we are left with only a cycle C_h and go to (6). Else let the unique cell be (i, j) and if it lies in a row $i\in I_h$ go to (3); if it lies in a column $j\in J'_h$ go to (4).

(6) Let (r, s) be the edge removed from C_h. Let $(i, j)=(r, s)$. Let $y_{ij}=\theta$. $F2=0$. Go to (7).

(7) Let $u=p_h(j)$. Set $y_{uj}=\beta_j-y_{ij}$ and set $i=u$.

(8) Let $v=p_h(i)$. If $v=\phi$, then let $v=s$. Set $y_{iv}=(\alpha_i-e_{ij}y_{ij})/e_{iv}$ and set $j=v$. If $(i, j)=(r, s)$ and $F2=0$ go to (9). If $(i, j)=(r, s)$ and $F2=1$, let $h=h+1$ and go to (1). Else go to (7).

(9) A linear expression for $y_{rs}=\theta$ is again obtained involving θ, α_i, β_j and e_{ij}. Solve for θ in terms of the known values of α_i, β_j and e_{ij}. Let $F2=1$. Replace y_{rs} with this newly found known value of θ and set $(i, j)=(r, s)$. Go to (7).

The values of y_{ij} determined above due to algorithm A6 lead to the following theorem.

THEOREM 6: For the operator dR_A, the optimum solution X^A is given by

$$x^A_{ij}=x_{ij}+\delta y_{ij} \quad \text{for } i\in I \text{ and } j\in J. \qquad (y_{m+1,n+1}\equiv 0). \tag{34}$$

The maximum extent to which this basis preserving operator can be applied is given by

$$\mu^A=\operatorname{Min}\begin{cases}(U_{ij}-x_{ij})/y_{ij} & \text{for } \{(i,j)\in B|y_{ij}>0\}\\ -x_{ij}/y_{ij} & \text{for } \{(i,j)\in B|y_{ij}<0\}\end{cases}. \tag{35}$$

PROOF: Let us first show that the solution X^A defined in (34) is basic, i.e., satisfies (6) and (7) for P^A. Since $y_{ij}=0$ for $(i, j)\in B$, $x^A_{ij}=x_{ij}$ for these cells. From (32), $\sum_{j\in J} e_{ij}y_{ij}=\alpha_i$ so that $\sum_{j\in J} e_{ij}x^A_{ij}=a_i+\delta\alpha_i=a^A_i$ for $i\in I$ so that (6) and (7) are satisfied for problem P^A. By the choice of μ^A in Equation (35), the solution X^A is primal feasible for all $0\leq\delta\leq\mu^A$. By theorem 3(c) the solution is dual feasible as well and hence optimal.

The quantity μ^A gives the maximum δ for δR_A to be basis preserving, since for $\delta>\mu^A$, x^A_{ij} corresponding to the leaving cell (e_1, f_1), (i.e., the cell at which the minimum of (35) is taken on) can be shown to violate (8).

ALGORITHM A7: For determining μ associated with a basis preserving rim operator and the leaving cell (e_1, f_1).

(1) If μ^A is to be determined (δR_A) go to (4). If $\mu^{\pm}$ is to be determined $(\delta R^{\pm}_{pq})$ go to (2).

(2) If $(p, q)\notin B$ go to (3). Otherwise determine $\mu^+(\mu^-)$ from (25), ((27)). Record $(e_1, f_1)=(p, q)$. STOP.

(3) Find the two-tree Γ when (p, q) is added to B. Using Algorithm 4(a) (4(b)) identify Γ_1 and Γ_2. Determine $\mu^+(\mu^-)$ from (29), ((31)). Record (e_1, f_1) as the cell at which the minimum of (29), ((31)) is taken on. STOP.

(4) Find $Y=\{y_{ij}\}$ by using Algorithm 6. Determine μ^A from (35). Record (e_1, f_1) as the cell at which the minimum of (35) is taken on. STOP.

ALGORITHM A8: For applying the basis preserving rim operator.

(1) If it is an area operator (δR_A) go to (4). If it is a cell operator $(\delta R^{\pm}_{pq})$ go to (2).

(2) Set the data of $P^{\pm}$ same as P except that $a^{\pm}_p=a_p\pm e_{pq}\delta$; $b^{\pm}_q=b_q\pm\delta$. Set $B^{\pm}=B$, $LB^{\pm}=LB$; $UB^{\pm}=UB$; $D^{\pm}=D$ and $Z^{\pm}=Z\pm\delta d_{pq}$. If $(p, q)\notin B$ go to (3). Otherwise determine $X^+(X^-)$ using (24), ((26)). STOP.

(3) Find the two-tree Γ obtained by adding (p, q) to the basis. Utilizing Algorithm 4(a), (4(b)) identify Γ_1 and Γ_2. Determine $X^+(X^-)$ by using (28), ((30)). STOP.

(4) Set the data of problem P^A same as P except that $a^A_i=a_i+\delta\alpha_i$ for $i\in I$, $b^A_j=b_j+\delta\beta_j$ for $j\in J'$.

u_i \ v_j →	31/5	28/5	3	4		a_i
0	3 | 15 | 3 | 15	2 | 6 | N	2 | 35 | (3) | 40	5 | 60 | (4) | 80	1 | 5 | (0) | N	300
-1/5	1 | 135 | (6) | 150	3 | 30 | (5) | 75	5 | 8 | 25	10 | 15 | N	1 | 0 | N	225
-13/10	4 | 20 | (1) | 60	2 | 30 | (3) | N	2 | 10 | 30	1 | 6 | 50	1 | 0 | N	140
0	1 | 100 | N	1 | 100 | N	1 | 100 | N	1 | 100 | N	1 | 1000 | (0) | N	1000
b_j	170	60	35	60	-	

FIGURE 4. (FIGURE 4 of Ref. [24])

Set $B^A = B$; $LB^A = LB$, $UB^A = UB$, $D^A = D$ and $Z^A = Z + \delta\left[\sum_{i\in I} \alpha_i u_i + \sum_{j\in J'} \beta_j v_j\right]$. Find $Y = \{y_{ij}\}$ using Algorithm 6. Find X^A using (34). STOP.

EXAMPLE (continued from [24]): Referring to Figure 4 for the operator $\delta R^+_{1,3}$, we find $\mu^+ = 40 - 35 = 5$ from (25). Similarly for $\delta R^+_{3,2}$, $\mu^+ = N - 30$. In this case since N is a large positive number, this implies that this operator can be applied to any large extent for cell (3, 2). On the other hand, the operator $\delta R^-_{2,2}$ can be applied to a maximum extent of $\mu^- = 30$, from (27). For instance, if $\delta = 10$, the optimum solution will have $x^-_{22} = 30 - 10 = 20$ with all other x_{ij}'s remaining the same (26). It can be easily verified that the solution is primal feasible with $a^-_2 = 225 - (3 \times 10) = 195$ and $b^-_2 = 60 - 10 = 50$. The solution is dual feasible and hence optimal. The optimal cost $Z^- = Z - \delta \times d_{22} = 1460 - (10 \times 5) = 1410$, which is true since the only change in the optimal cost is due to the decrease of x_{22} at the rate of $c_{22} = d_{22} = 5$.

D: $\{d_{ij} = \theta_{ij} u_i + v_j\}$

d_{ij} GIVEN IN THE CENTER AND θ_{ij} GIVEN IN NORTHWEST CORNER OF A CELL (i,j) $i\in I$; $j\in J'$:

u_i \ v_j →	31/5	28/5	3	4
0	3 | 31/5	2 | 28/5	2 | 3	3 | 4
-1/5	1 | 6	3 | 5	5 | 2	10 | 2
-13/10	4 | 1	2 | 3	2 | 2/5	1 | 27/10
0	1 | 31/5	1 | 28/5	1 | 3	1 | 4

FIGURE 6. (FIGURE 6 of Ref. [24])

From Figure 6 [24] we see that $d_{23} = 2$ which means that the total cost will *increase* at the rate of 2 if a_2 and b_3 were simultaneously *increased.*

Let us apply the operator $\delta R_{23}^{\ddagger}$ utilizing algorithm 4(a). The following gives the steps of Algorithm 4(a) and the results when these are implemented, where $\Gamma = \{(2, 2), (3, 2), (3, 1), (2, 1)\} \cup \{2, 3\} \cup \{(1, 3), (1, 5)\}$.

Steps of A4(a) *Result due to the application of this step*

(0) $(e, f) = (2, 3) \in \Gamma_1$, $x = -1$ and $m_{23} = -1$; let $(u, v) = (2, 3)$
$F1 = 0$ (row segment). All $m_{ij} = 0$ for $(i, j) \in \Gamma$
$(r, s) = (2, 2)$.

(1) $d(r_2) = 0$, go to (5).

(5) $M = \{r_2, r_3, c_1, c_2\}$; $c_2 = p(r_2)$; $x = +\frac{6}{5} \times \frac{5}{3} = 2$;
$m_{2,2} = 0 + 2$; $(2, 2) \in \Gamma_1$; $M = M - \{c_2\}$; $v = c_2$.

(6) $r_3 = p(c_2)$; $x = -2$; $m_{32} = 0 - 2$; $(3, 2) \in \Gamma_2$;
$M = M - \{r_3\}$; $u = r_3$.

(7) $c_1 = p(r_3)$; $x = +2 \times \frac{2}{4} = +1$; $m_{31} = 0 + 1$; $(3, 1) \in \Gamma_1$
$M = M - \{c_1\}$; $v = c_1$. Go to (6).

(6) $r_2 = p(c_1)$; $x = -1$; $m_{21} = 0 + (-1)$; $(2, 1) \in \Gamma_2$
$M = M - \{(r_2\} = \emptyset$; $u = r_2$.

(7) $M = \emptyset$ and $F1 = 0$, so $F1 = 1$; $(u, v) = (2, 3)$; $x = -1$
go to (3).

(3) Go to 4.

(4) $r_1 = p(c_3)$; $x = +1$; $m_{1,3} = 0 + 1$; $(1, 3) \in \Gamma_1$; $u = r_1$
go to (1).

(1) $d(r_1) = 0$. Go to 5.

(5) $M = \{r_1\}$; $n + 1 = 5 = p(r_1)$; $m_{1,5} = 0 - 2$. Since
$F1 = 1$ go to 11.

(11)

$$\mu^+ = \text{Min} \begin{cases} \dfrac{75-30}{2}, \dfrac{60-20}{1}, \dfrac{40-35}{1} & \text{amounts} \\ (2, 2), \quad (3, 1), \quad (1, 3) & \text{cells in } \Gamma_1 \\ \dfrac{30}{2}, \quad \dfrac{135}{1}, \quad \boxed{\dfrac{5}{2}} & \text{amounts} \\ (3, 2), \quad (2, 1), \quad (1, 5) & \text{cells in } \Gamma_2 \end{cases}$$

$\mu^+ = 5/2 = 2.5$ and cell $(1, 5) \in B$ reaches the zero level.
$(e_1, f_1) = (1, 5)$.

Thus the maximum extent that this operator can be applied is $\mu^+ = 5/2$ to maintain basis preserving status. Total cost due to this increases at the rate of $c_{22} + c_{31} + c_{13} = 5 + 1 + 3 = 11$ (on the getter cells)

ut decreases at the rate $c_{32}+c_{21}+c_{15}=3+6+0=9$ (on the given cells) so that the net *increase* is t the rate of 2. (Note $d_{23}=2$). We find $\mu^+ = 5/2$ limited by the cell (1, 5) reaching its lower bound.

δR_{23}^+ (P)

u_i \ v_j	31/5	28/5	3	4	–	a_i
0	3 [3] 15 [15]	6 [2] [N]	35+δ [2] (3) [40]	(4) 60 [3] [80]	(0) 5−2δ [1] [N]	300
−1/5	(6) 135−δ [1] [150]	30+2δ (5) [3] [75]	(8) [5] [25]	15 [10] [N]	0 [1] [N]	225 +5δ
−13/10	(1) 20+δ [4] [60]	30−2δ (3) [2] [N]	10 [2] [30]	6 [1] [50]	0 [1] [N]	140
0	100 [1] [N]	100 [1] [N]	100 [1] [N]	100 [1] [N]	(0) 1000 [1] [N]	1000
b_j	170	60	35+δ	60	–	

$\mu^+ = 2.5$ AT (1,5)

FIGURE 7.

We now consider an area operator δR_A for which α_i and β_j are shown on the rims of Figure 9. The alues of $Y=\{y_{ij}\}$ determined by algorithm A6 are as follows.

$2R_{23}^+$ (P)

u_i \ v_j	31/5	28/5	3	4	–	a_i
0	3 [3] 15 [15]	6 [2] [N]	(3) 37 [2] [40]	(4) 60 [3] [80]	(0) 1 [1] [N]	300
−1/5	(6) 133 [1] [150]	(5) 34 [3] [75]	9 [5] [25]	15 [10] [N]	0 [1] [N]	235
−13/10	(1) 22 [4] [60]	(3) 26 [2] [N]	10 [2] [30]	6 [1] [50]	0 [1] [N]	140
0	100 [1] [N]	100 [1] [N]	100 [1] [N]	100 [1] [N]	(0) 1000 [1] [N]	1000
b_j	170	60	37	60	–	

FIGURE 8.

The following gives the steps of algorithm A6 and the result due to implementation of these steps r α_i $(i \in I)$ and β_j $(j \in J')$ given as rims in Figure 9.

y_{ij} ARE GIVEN ADJACENT TO c_{ij} IN ▱,

FOR THE GIVEN SET OF α_i AND β_j

u_i \ v_j	31/5	28/5	3	4		a_i	α_i
0	3 | 15 | 15	6 | N	(3) 2 | 35 | 40	(4) -7 | 60 | 80	(0) 15 | 5 | N	300	-2
-1/5	(6) 31/5 | 135 | 150	(5) -7/5 | 30 | 75	8 | 25	15 | N	0 | N	225	+2
-13/10	(1) -6/5 | 20 | 60	(3) 22/5 | 30 | N	10 | 30	6 | 50	0 | N	140	4
0	100 | N	100 | N	100 | N	100 | N	(0) 1000 | N	1000	0
b_j	170	60	35	60	-	-	
β_j	5	3	2	-7			

FIGURE 9.

Steps of A6. *Result due to application of this step*

(0) $B = \{B_1, B_2, B_3\}$ corresponding to T_1, T_2, T_3,
$L = \{1, 2, 3\}$ $h = 1$.

(1) $r_1 =$ root of T_1; $(1, 5) \in B_1$ where $B_1 = \{(1, 5), (1, 4), (1, 3)\}$
F1 = 0. Go to (2).

(2) c_3 is a line with (1, 3) being unique in c_3. Go to (4).

(4) $y_{13} = \beta_3 = 2$; $\alpha_1 = \alpha_1 - e_{13}y_{13} = -2 - 4 = -6$;
$B_1 = B_1 - \{(1, 3)\}$ and $J_1' = \{3, 4\} - \{3\} = \{4\}$. Go to (2).

(2) c_4 is a line with (1, 4) being unique in c_4. Go to (4).

(4) $y_{14} = \beta_4 = -7$; $\alpha_1 = \alpha_1 - e_{14}y_{14} = -6 + 21 = 15$
$B_1 = B_1 - \{(1, 4)\}$; $J_1' = \{4\} - \{4\} = \phi$. Go to (2).

(2) r_1 is a line with (1, 5) being unique in r_1. Cell is
in col. $n + 1$ so $y_{1,5} = \alpha_1 = 15$; $h = 2$; go to (1).

(1) r_2 is root of T_2. F1 = 1. Go to (5).

(5) No unique cell in $B_2 = \{(2, 1), (2, 2), (3, 2), (3, 1)\}$.
Go to (6).

(6) (2, 2) is the edge removed from C_2. $(i, j) = (2, 2)$.
Let $y_{22} = \theta$; F2 = 0. Go to (7).

(7) $r_3 = p_2(c_2)$; $y_{32} = \beta_2 - \theta = 3 - \theta$; $i = r_3$.

(8) $c_1 = p_2(r_3)$; $y_{31} = [\alpha_3 - e_{32}y_{32}]/e_{31} = \dfrac{4 - 2(3 - \theta)}{4}$
$= 1 - 1/2(3 - \theta) = -1/2 + \theta/2$; $j = c_1$; Go to (7).

(7) $r_2 = p_2(c_1)$; $y_{21} = \beta_1 - y_{31} = 5 - (-1/2 + \theta/2)$
$= 11/2 - \theta/2$; $i = r_2$.

(8) $c_2 = p_2(r_2)$; $y_{22} = [\alpha_2 - e_{21}y_{21}]/e_{22} = [2 - 11/2 + \theta/2]/3. = -7/6 + \theta/6$; $u = c_2$; $(i, j) = (2, 2) = (r, s)$ and $F2 = 0$. Go to (9).

(9) $\theta = -7/6 + \theta/6$ or $\theta = -7/5$; $F2 = 1$; $y_{22} = -7/5$, $(i, j) = (2, 2)$, go to (7).

(7) $r_3 = p_2(c_2)$; $y_{32} = 3 - (-7/5) = 22/5$; $i = r_3$.

(8) $c_1 = p_2(r_3)$; $y_{31} = -1/2 + (-7/10) = -6/5$; $j = c_1$; Go to (7).

(7) $r_2 = p_2(c_1)$; $y_{21} = 11/2 - (-7/10) = 31/5$; $i = r_2$.

(8) $c_2 = p_2(r_2)$; $y_{22} = -7/6 + (-7/30) = -7/5$; $j = c_2$, $(i, j) = (2, 2) = (r, s)$ and $F2 = 1$; let $h = 3$. Go to (1).

(1) r_4 is the root of T_3; $(4, 5) \epsilon B_3$; $F1 = 0$; go to (2).

(2) r_4 is a line with unique cell ϵB_3. This cell is col. $n+1$ and so $y_{4,5} = \alpha_4 = 0$; $h = 4$; go to (1).

(1) $h = 4$; STOP.

$2R_A$

u_i \ v_j	31/5	28/5	3	4	–	a_i	α_i
0	[3] 15 → 3 → [15]	[2] 6 → [N]	[2] 39 → (3) → [40]	[3] 46 → (4) → [80]	[1] 35 → (0) → [N]	296	−2
−1/5	[1] 737/5 → (6) → [150]	[3] 136/5 → (5) → [75]	[5] 8 → [25]	[10] 15 → [N]	[1] 0 → [N]	229	+2
−13/10	[4] 88/5 → (1) → [60]	[2] 194/5 → (3) → [N]	[2] 10 → [30]	[1] 6 → [50]	[1] 0 → [N]	148	4
0	[1] 100 → [N]	[1] 100 → [N]	[1] 100 → [N]	[1] 100 → [N]	[1] 1000 → (0) → [N]	1000	0
d_j	180	66	39	46	–		
β_j	5	3	2	−7			

FIGURE 10.

These values of y_{ij} for $(i, j) \epsilon B$ due to algorithm $A6$ are shown adjacent to c_{ij} in each cell of Figure 9 ($y_{ij} = 0$ for the non-basic cells). From (34) and using the x_{ij} values of Figure 4 which are incorporated in Figure 7, we get from (35) $\mu^A = 75/31$, this time being limited by cell (2, 1). Application of the operator R_A $(2 < 75/31)$ results in the optimal solution of Figure 10. To find the cost effect of the operator we compute $\sum_{i \epsilon I'} \alpha_i u_i + \sum_{j \epsilon J'} \beta_j v_j = 89/15$ so that from Theorem 3(c) $Z^A = Z + (2 \times 89/15) = 1460 + (178/15) = 22078/15$ which checks with the total cost $\sum_{i \epsilon I} \sum_{j \epsilon J} c_{ij} x_{ij}^A$ computed from Figure 10.

In this problem all the μ's turn out to be strictly positive. It is easy to infer from (25), (27), (29), (31), (35) that this is not necessarily the case if the solution is (primal) degenerate i.e., x_{ij} corresponding to some basic cell (i, j) takes on either its lower or upper bound.

3. BASIS-PRESERVING COST OPERATORS

We now consider cost operators that transform the optimal solutions when only the cost elements

c_{ij} are changed in such a way that the optimal basis is preserved.

THEOREM 7: For cost operators the primal solutions are determined as follows:

(a) For the operator δC^+_{pq}, $X^+ = X$ and $Z^+ = Z + \delta x_{pq}$.

(b) For the operator δC^-_{pq}, $X^- = X$ and $Z^- = Z - \delta x_{pq}$.

(c) For the operator δC_A; $X^A = X$ and

$$Z_A = Z + \delta\left[\sum_{i\epsilon I}\sum_{j\epsilon J'} \gamma_{ij} x_{ij}\right].$$

PROOF: (a) Since the operator is basis preserving, $LB^+ = LB$, and $UB^+ = UB$ so that $x^+_{ij} = x_{ij}$ for these nonbasic cells. Since it is a cost operator $a^+_i = a_i$ for $i\epsilon I$ and $b^+_j = b_j$ for $j\epsilon J'$. Thus it follows that $x^+_{ij} = x_{ij}$ for $(i, j)\,\epsilon B = B^+$. This proves that $X^+ = X$. Now $Z^+ = \sum_{i\epsilon I}\sum_{j\epsilon J} c^+_{ij} x^+_{ij} = \sum_{i\epsilon J}\sum_{j\epsilon J} c^+_{ij} x_{ij}$. But by definition of δC^+_{pq}, $c^+_{ij} = c_{ij}$ for $(i, j) \neq (p, q)$ and $c^+_{pq} = c_{pq} + \delta$, so that

$$Z^+ = \sum_{i\epsilon I}\sum_{j\epsilon J} c_{ij} x_{ij} + \delta x_{pq} = Z + \delta x_{pq}.$$

The proofs of part (b) and (c) are similar.

Theorems 8–10 determine the optimal dual solutions to the problems obtained after applying the cost operators of different types. Let us consider first cell operators with $(p, q) \notin B$.

THEOREM 8: (a) For the operator δC^+_{pq} with $(p, q) \notin B$

$D^+ = D$ and

$$\mu^+ = \begin{cases} M & \text{if } (p, q)\,\epsilon LB \\ u_p e_{pq} + v_q - c_{pq} & \text{if } (p, q)\,\epsilon UB. \end{cases} \tag{36}$$

(b) For the operator δC^-_{pq} with $(p, q) \notin B$, $D^- = D$ and

$$\mu^- = \begin{cases} c_{pq} - u_p e_{pq} - v_q & \text{if } (p, q)\,\epsilon LB \\ M & \text{if } (p, q)\,\epsilon UB, \end{cases} \tag{37}$$

where M is an arbitrarily large positive number.

PROOF: (a) Since $B^+ = B$ and $c^+_{ij} = c_{ij}$ for $(i, j)\,\epsilon B$, from (20) it can be seen that $u^+_i = u_i$ for $i\epsilon I$ and $v^+_j = v_j$ for $j\epsilon J'$ so that $D^+ = D$. For each $(i, j) \notin B$, with $(i, j) \neq (p, q)$, $c^+_{ij} = c_{ij}$ and the duals u_i and v_j are unaltered so that the dual feasibility condition (18) or (19) is satisfied, since it is true for problem P. As far as the cell (p, q) is concerned, two cases arise:

(i) $(p, q)\,\epsilon LB$: $c^+_{pq} - u^+_p e_{pq} - v^+_q = c_{pq} + \delta - u_p e_{pq} - v_q \geq 0$ since $\delta \geq 0$; and $c_{pq} - u_p e_{pq} - v_q \geq 0$ since the solution is dual feasible for P. As there is no upper bound for δ, we impose an arbitrary bound of M.

(ii) $(p, q)\,\epsilon UB$. $c^+_{pq} - u^+_p e_{pq} - v^+_q = c_{pq} + \delta - e_{pq} u_p - v_q \leq 0$ provided $\delta \leq e_{pq} u_p + v_q - c_{pq} = \mu^+$. For $\delta > \mu^+$, the cell (p, q) would lose dual feasibility.

Thus for $0 \leq \delta \leq \mu^+$, the solution $D^+ = D$ is dual feasible. Since $X^+ = X$ is primal feasible due to no change of rim conditions, the solution is optimal. The proof of (b) is similar.

For operator δC^+_{pq} with $(p, q)\,\epsilon B$ we need to introduce the following definition to analyze the changes

in u_i and v_j. Let $(p, q) \epsilon B$ be contained in one of the component T_k where $G(T_k) = \{I_k \cup J_k, B_k\}$. B_k can be partitioned as C_k and C_k^* such that $C_k^* = B_k - C_k$. The cells ϵC_k are the cycle elements while the cells ϵC_k^* are called *branches*.

REMARK 14: Let $(p, q) \epsilon B_k$ and $\Omega_k = B_k - \{(p, q)\}$. The following mutually exclusive and collectively exhaustive cases are relevant.

(1) $(p, q) \epsilon C_k \subseteq B_k$ and predecessor of $p = q$.

(2) $(p, q) \epsilon C_k \subseteq B_k$ and predecessor of $q = p$.

(3) $(p, q) \epsilon C_k^* \subset B_k$ and predecessor of $p = q$.

(4) $(p, q) \epsilon C_k^* \subset B_k$ and predecessor of $q = p$.

DEFINITION 17: The sets I_p and I_{qk} defined below (38), partition I_k such that $I_p \cup I_{qk} = I_k$ and $I_p \cap I_{qk} = \emptyset$. Similarly J'_{pk} and J'_q partition J'. From these the sets I_q and J_p are also defined.

$$
\begin{aligned}
I_p &= \{p\} \cup \{i \epsilon I \mid i \text{ is connected to } p \text{ in } \Omega_k\} \\
I_{qk} &= I_k - I_p \\
I_q &= I - I_p \\
J'_q &= \{q\} \cup \{j \epsilon J' \mid j \text{ is connected to } q \text{ in } \Omega_k\} \\
J'_{pk} &= J'_k - J'_q \\
J'_p &= J' - J'_q.
\end{aligned} \tag{38}
$$

It is easy to see that, in cases (1) and (2) where $(p, q) \epsilon C_k$, $I_p = I_k$, so that $I_q = I - I_k$ and $J'_q = J'_k$ so that $J'_p = J' - J'_k$. In cases (3) and (4) where $(p, q) \epsilon C_k^*$, I_{qk} or J_{pk} need not be empty. Moreover, I_p and J'_q are never empty. An algorithm for finding the sets I_p, I_{qk}, I_q, J'_q, J'_{pk}, J'_p is given in [4]. This is based on the "scanning routine" of Remark 9 in [21].

The following two algorithms A9(a) and A9(b) provide a method for finding the duals u_i and v_j for $i, j \epsilon S(T_k)$ when the operator $\delta C^+_{(e,f)}$ is applied $((e, f) = (p, q))$ when $(e, f) \epsilon C_k \subseteq B_k$ or when $(e, f) \epsilon C_k^* \subset B_k$ respectively. For the case of $\delta C^-_{(e,f)}$ the same algorithms are applicable since we use $-m_i$ and $-n_j$ as the coefficients of δ in (41). Further the dual variables u_i and v_j are unaltered for $i \epsilon I_q$ and $j \epsilon J'_p$.

ALGORITHM A9(a): For finding the dual variables u_i, v_j when the operator δC^+_{ef} is applied given that $(e, f) \epsilon C_k \subseteq B_k$ (Case a). (Note $(e, f) = (p, q)$.)

(0) Denote C_k by C and C_k^* by C^*. Let $(i, j) = (e, f)$. If $p(i) = j$ set $n_j = (1 - \rho)$ (note: $(1 - \rho) \leqslant 0$) $v_j = v_j + n_j \delta$ and go to (1). If $p(j) = i$ set $m_i = (1 - \rho)/e_{ij}$ (note: $m_i \leqslant 0$), $u_i = u_i + m_i \delta$ and go to (2).

(1) Let $a = p(j)$. Find $m_a = -n_j/e_{aj}$. Let $u_a = u_a + m_a \delta$. If $a = e$ go to (3). Else let $i = a$.

(2) Let $b = p(i)$. Find $n_b = -e_{ib} m_i$. Let $v_b = v_b + n_b \delta$. If $b = f$, go to (3). Else let $j = b$ and go to (1).

(3) (Evaluation of u_i, v_j for cells C^*). Let $t = 1$.

(i) Find $D_t = \{w \mid w \epsilon S(C^*) \text{ and } d(w) = t\}$. If $D_t = 0$, STOP. Else go to 3 (ii).

(ii) For every $w \epsilon D_t$, if row $i = p(w)$; find $n_w = -e_{iw} m_i$ and set $v_w = v_w + n_w \delta$. Go to (3) (iii). If col $j = p(w)$, find $m_w = -n_j/e_{wj}$ and set $u_w = u_w + m_w \delta$.

(iii) Set $t = t + 1$ and go to (3) (i).

It is to be noted that the value of $n_f = (1 - \rho)$ when $p(e) = f$ and $m_e = (1 - \rho)/e_{ef}$ when $p(f) = e$ are obtained by arbitrarily setting n_f (or m_e) as λ, traversing the cycle once and solving for λ. The reader could easily verify these two results.)

REMARK 15: Consider the operator $\delta C^{\pm}_{ef}$ for case (b) where $(e, f) \epsilon C^*_k \subset B_k$. Two cases are relevant for developing algorithm A9(b):

(i) $d(e) < d(f)$ (i.e., $p(f) = e$), and

(ii) $d(f) < d(e)$ (i.e., $p(e) = f$). It is necessary to partition the set of nodes $I_k \cup J'_k$ into two sets S_1 and S_2 as follows:

For case (i), $S_1 = \{w \mid w \epsilon I_k \cup J'_k$ and connected to e in $\Omega_k\}$ and for case (ii), $S_1 = \{w \mid w \epsilon I_k \cup J'_k$ and connected to f in $\Omega_k\}$ with $S_2 = \{I_k \cup J_k'\} - S_1$. In case (i) u_e and every dual variable corresponding to a node ϵS_1 do not change while v_f and every dual variable for a node ϵS_2 change. Similarly in case (ii), v_f and every dual variable corresponding to a node ϵS_1 do not change while u_e and every dual variable for a node ϵS_2 change due to the application of the operator $\delta C^{\pm}_{ef}$. These changes are given in Algorithm A9(b). The knowledge of whether C_k is a loop or cycle does not make any difference. Here, also, we can directly give the value for $n_f = 1$ for case (i) and $m_e = 1/e_{ef}$ in case (ii). These are used in Algorithm A9(b).

ALGORITHM A9(b): For finding the dual variables u_i and v_j when the operator δC^{+}_{ef} is applied given that $(e,f) \epsilon C_k^* \subset B_k$ (case b). (Note $(e,f) = (p, q)$).

(0) Denote C_k by C and C^*_k by C^*. Let $(i, j) = (e, f)$. If $p(i) = j$, set $m_i = 1/e_{ij}$, $u_i = u_i + m_i\delta$. Let $t = d(i) + 1$, $SR = \{i\}$ and go to (1). If $p(j) = i$, set $n_j = 1$, $v_j = v_j + n_j\delta$. Let $t = d(j) + 1, SC = \{j\}$ and go to (3).

(1) Find $SC = \{j \mid d(j) = t$ and $p(j) = i$ for every $i \epsilon SR\}$. If $SC = \emptyset$ go to (5). Else go to (2).

(2) For each $i \epsilon SR$ and each $j \epsilon SC$ let $n_j = -e_{ij}m_i$ and set $v_j = v_j + n_j\delta$. Let $t = t + 1$.

(3) Find $SR = \{i \mid d(i) = t$ and $p(i) = j$ for every $j \epsilon SC\}$. If $SR = \emptyset$ go to (5). Else go to (4).

(4) For each $i \epsilon SR$ and each $j \epsilon SC$ let $m_i = n_j/e_{ij}$ and set $u_i = u_i + m_i\delta$. Let $t = t + 1$ and go to (1).

(5) STOP.

THEOREM 9: (a) For δC^{+}_{pq} with $(p, q) \epsilon B$, the optimal dual solutions are given below:

(i) If $(p, q) \epsilon C_k$ (i.e., $d(p) = d(q) = 0$)

$$u_i^+ = \begin{cases} u_i + m_i\delta & \text{for } i \epsilon I_p \\ u_i & \text{for } i \epsilon I_q \end{cases}$$

$$v_j^+ = \begin{cases} v_j + n_j\delta & \text{for } j \epsilon J'_q \\ v_j & \text{for } j \epsilon J'_p; \end{cases} \tag{39'}$$

(ii) if $(p, q) \epsilon C^*_k$ and $d(v) < d(q)$ (case (i) of Rem. 15)

$$u_i^+ = \begin{cases} u_i + m_i\delta & \text{for } i \epsilon I_{qk} \\ u_i & \text{for } i \epsilon \bar{I}_{qk} \text{ (where } \bar{I}_{qk} = I - I_{qk}) \end{cases}$$

$$v_j^+ = \begin{cases} v_j + n_j\delta & \text{for } j \epsilon J'_q \\ v_j & \text{for } j \epsilon J'_p; \end{cases} \tag{39''}$$

(iii) if $(p, q)\epsilon C_k^*$ and $d(p) > d(q)$ (case (ii) of Rem. 15)

$$u_i^+ = \begin{cases} u_i + m_i\delta & \text{for } i\epsilon I_p \\ u_i & \text{for } i\epsilon I_q \end{cases}$$

(39''')

$$v_j^+ = \begin{cases} v_j + n_j\delta & \text{for } j\epsilon J'_{pk} \\ v_j & \text{for } j\epsilon \bar{J}_{pk} (\text{where } \bar{J}'_{pk} = J' - J'_{pk}). \end{cases}$$

where m_i and n_j are functions of e_{ij} as given in A9(a) and A9(b) as the case may be and the sets I_p, I_{qk}, I_q, I'_q, J'_{pk} and J'_p are given in (38). The maximum extent μ^+ to which this basis preserving operator can be applied is given below:

(i) If predecessor of $p = q$ and $(p, q)\epsilon C_k$

$$\mu^+ = \text{Min} \begin{cases} (e_{ij}u_i + v_j - c_{ij})/|n_j| & \text{for } (i, j)\epsilon[(I_q \times J'_q) \cap UB] \\ (c_{ij} - e_{ij}u_i - v_j)/e_{ij}m_i & \text{for } (i, j)\epsilon[(I_p \times J_p) \cap LB] \end{cases}$$

(40) and if $(p, q)\epsilon C_k^* (d(p) > d(q))$

$$\mu^+ = \text{Min} \begin{cases} (e_{ij}u_i + v_j - c_{ij})/|n_j| & \text{for } (i, j)\epsilon[(I_q \times J'_{pk}) \cap UB] \\ (c_{ij} - e_{ij}u_i - v_j)/e_{ij}m_i & \text{for } (i, j)\epsilon[I_p \times \bar{J}_{pk}) \cap LB] \end{cases}$$

where $\bar{J}_{pk} = (J - J'_{pk})$.

(ii) If predecessor $q = p$ and $(p, q)\epsilon C_k$

$$\mu^+ = \text{Min} \begin{cases} (e_{ij}u_i + v_j - c_{ij})/e_{ij}|m_i| & \text{for } (i, j)\epsilon[(I_p \times J'_p) \cap UB] \\ (c_{ij} - e_{ij}u_i - v_j)/n_j & \text{for } (i, j)\epsilon[(I_q \times J'_q) \cap LB] \end{cases}$$

(41) and if $(p, q)\epsilon C_k^* (d(p) < d(q))$

$$\mu^+ = \text{Min} \begin{cases} (e_{ij}u_i + v_j - c_{ij})/e_{ij}|m_i| & \text{for } (i, j)\epsilon[(I_{qk} \times J'_p) \cap UB] \\ (c_{ij} - e_{ij}u_i - v_j)/n_j & \text{for } (i, j)\epsilon[(\bar{I}_{qk} \times J'_q) \cap LB]. \end{cases}$$

(b) For $\delta C_{\bar{p}q}$ with $(p, q)\epsilon B$ the optimal dual solutions are the following:
(i) If $(p, q)\epsilon C_k$ (i.e., $d(p) = d(q) = 0$)

$$u_{\bar{i}} = \begin{cases} u_i - m_i\delta & \text{for } i\epsilon I_p \\ u_i & \text{for } i\epsilon I_q \end{cases}$$

(42′)

$$v_j^- = \begin{cases} v_j - n_j\delta & \text{for } j \in J'_q \\ v_j & \text{for } j \in J'_p. \end{cases}$$

(ii) If $(p, q) \in C_k^*$ and $d(p) < d(p)$ (case (i) of Rem. 15)

$$u_i^- = \begin{cases} u_i - m_i\delta & \text{for } i \in I_{qk} \\ u_i & \text{for } i \in \bar{I}_{qk} (\bar{I}_{qk} = I - I_{qk}) \end{cases}$$

(42″)

$$v_j^- = \begin{cases} v_j - n_j\delta & \text{for } j \in J'_q \\ v_j & \text{for } j \in J'_p. \end{cases}$$

(iii) If $(p, q) \in C_k^*$ and $d(p) > d(q)$ (case (ii) of Rem. 15)

$$u_i^- = \begin{cases} u_i - m_i\delta & \text{for } i \in Ip \\ u_i & \text{for } i \in I_q \end{cases}$$

(42‴)

$$v_j^- = \begin{cases} v_j - n_i\delta & \text{for } j \in J'_{pk} \\ v_j & \text{for } j \in \bar{J}'_{pk} (\bar{J}'_{pk} = J' - J'_{pk}). \end{cases}$$

The maximum extent μ^- to which the basis-preserving operator can be applied is given below:
(i) If predecessor of $p = q$ and $(p, q) \in C_k$

$$\mu^- = \text{Min} \begin{cases} (c_{ij} - e_{ij}u_i - v_j)/|n_j| & \text{for } (i, j) \in [(I_q \times J'_q) \cap LB] \\ (e_{ij}u_i + v_j - c_{ij})/e_{ij}m_i & \text{for } (i, j) \in [(I_p \times J'_p) \cap UB] \end{cases}$$

(43) and if $(p, q) \in C_k^* (d(p) > d(q))$

$$\mu^- = \text{Min} \begin{cases} (c_{ij} - e_{ij}u_i - v_j)/|n_j| & \text{for } (i, j) \in [(I_q \times J'_{pk}) \cap LB] \\ (e_{ij}u_i + v_j - c_{ij})/e_{ij}m_i & \text{for } (i, j) \in [(I_q \times \bar{J}'_{pk}) \cap UB] \end{cases}$$

(ii) If predecessor of $q = p$ and $(p, q) \in C_k$

$$\mu^- = \text{Min} \begin{cases} (c_{ij} - e_{ij}u_i - v_j)/e_{ij}|m_i| & \text{for } (i, j) \in [(I_p \times J_p) \cap LB] \\ (e_{ij}u_i + v_j - c_{ij})/n_j & \text{for } (i, j) \in [(I_q \times J'_q) \cap UB] \end{cases}$$

(44) and if $(p, q) \epsilon C_k^*$ $(d(p) < d(q))$.

$$\mu^- = \text{Min} \begin{cases} (c_{ij} - e_{ij}u_i - v_j)/e_{ij}|m_i| & \text{for } (i,j) \epsilon [(I_{qk} \times J'_p) \cap LB] \\ (e_{ij}u_i + v_j - c_{ij})/n_j & \text{for } (i,j) \epsilon [(\bar{I}_{qk} \times J'_q) \cap UB]. \end{cases}$$

(In (40) and (44), $J_p = J'_p \cup \{(n+1)\}$.)

The minimum in (40) or (41) and (43) or (44) are set equal to M if the relevant sets over which the minimization is done is empty.

PROOF: (a) We will prove the case when $(p, q) \epsilon C_k$ and leave the proof for the case $(p, q) \epsilon C_k^*$ since it is similar.

We note that $e_{ij}u_i^+ + v_j^+ = e_{ij}u_i + v_j$ for $(i, j) \epsilon (I_q \times J'_p)$ since all the basic cells in $B - B_k$ are in $I_q \times J'_p$ by definition 17 and remark 14. Moreover the algorithms change u_i for $i \epsilon I_p$ and v_j for $j \epsilon J'_q$ only. Since $c_{ij}^+ = c_{ij}$ for $(i, j) \neq (p, q)$, Equation (17) is satisfied for all the basic cells in $(I_q \times J'_p)$ for problem P^+. Thus Equation (18) and (19) are satisfied for cells $(i,j) \epsilon (I_q \times J'_p) \cap LB$ and $(i,j) \epsilon (I_q \times J'_p) \cap UB$, respectively, for problem P^+. Based on the algorithms A9(a) and A9(b), m_i and n_j are computed from basic cells $(i,j) \epsilon (I_p \times J'_q)$. Hence for cells $(i,j) \epsilon \Omega_k$, $c_{ij}^+ = c_{ij}$ and it is easy to verify that (17) is satisfied for cells $(i, j) \epsilon (I_p \times J'_q) - \{(p, q)\}$. For the basic cell (p, q), $c_{pq}^+ = e_{pq}(u_p + m_p\delta) + v_q + n_q\delta = (e_{pq}u_p + v_q) + (e_{pq}m_p + n_q)\delta = (e_{pq}u_p + v_q) + 1\delta = c_{pq} + \delta$. Thus solution (39) satisfies (19) for P^+.

For the nonbasic cells $(i,j) \epsilon (I_q \times J'_p)$, $c_{ij}^+ = c_{ij}$ and $e_{ij}u_i^+ + v_j^+ = e_{ij}u_i + v_j$ (since $u_i^+ = u_i$ and $v_j^+ = v_j$) so that the dual feasibility conditions (18) and (19) are satisfied.

For $(i, j) \epsilon (I_p \times J'_p)$, $e_{ij}u_i^+ + v_j^+ = e_{ij}(u_i + m_i\delta) + v_j$ and for $(i, j) \epsilon (I_q \times J'_p)$ $e_{ij}u^+_i + v^+_j = e_{ij}u_i + v_j + n_j\delta$. For (i, j) which are nonbasic but $\epsilon (I_p \times J'_q)$, $e_{ij}u_i^+ + v_j^+ = e_{ij}(u_i + m_i\delta) + (v_j + n_j\delta) = e_{ij}u_i + v_j + (e_{ij}m_i + n_j)$. But $(e_{ij}m_i + n_j) = 0$ by algorithm A9(a), A9(b) for these cells. Thus the dual feasibility conditions (18) and (19) are satisfied for nonbasic $(i, j) \epsilon (I_p \times J'_q)$.

We know that in cases (1) and (3) for $i \epsilon I_p$, $u_i^+ = u_i + m_i\delta$, but m_i is positive while for $j \epsilon J'_q$, $v_j^+ = v_j + n_j\delta$ with n_j being negative. The reverse is true for cases (2) and (4). Thus for cases (1) and (3) for nonbasic $(i, j) \epsilon (I_p \times J_p)$ only the cells in the lower bound may become infeasible if δ is increased indefinitely, whereas $(i, j) \epsilon (I_p \times J'_p) \cap UB$ will *a fortiori* be contained in UB with no limit for δ. Similarly, only cells $(i, j) \epsilon (I_q \times J'_q) \cap UB$ may become infeasible if δ is increased indefinitely, but cells $(i, j) \epsilon (I_q \times J'_q) \cap LB$ have no effect. A similar argument can be given to cases (2) and (4). Thus the maximum extent μ^+ for cases (1) and (3) or cases (2) and (4) is the expression given in Equation (40) or (41), respectively.

(b) Proof for part (b) is similar.

REMARK 16: Due to primal nondegeneracy assumption for the basic cell (p, q), $0 < x_{pq} < U_{pq}$. If $\mu^+ = M$ this means that regardless of how large the cost c_{pq} may be, the optimal solution continues to assign positive machine hours of p to product q. In other words there is no primal feasible solution to the problem with $x_{pq} = 0$. Similarly we would expect x_{pq} to increase when c_{pq} is decreased. Thus the case $\mu^- = M$ would mean that x_{pq} has already reached its maximum value $\left(x_{pq} = \min\left(\frac{a_p}{e_{pq}}, b_q, U_{pq}\right)\right)$. Both these results can be proved more rigorously by a study of the structure of cells in B, LB, and UB for the cases when the sets over which the minimization is done turn out to be empty. If the sets over

which minimization is done is empty for determining $\mu^+(\mu^-)$, then $\mu^+(\mu^-)$ is set equal to M.

Let us now consider the area cost operator δC_A, for which the cost entry c_{ij} becomes $c_{ij}+\delta\gamma_{ij}$ for $i \epsilon I', j \epsilon J'$. To study the effect of the cost changes on the dual variables, we define u_i^* for $i \epsilon I'$ and v_j^* for $j \epsilon J'$, such that

$$e_{ij}u_i^* + v_j^* = \gamma_{ij} \qquad \text{for } (i,j) \epsilon B; \tag{45}$$

i.e., u_i^* and v_j^* are determined from γ_{ij} in the same manner as u_i and v_j are computed from c_{ij}. However, γ_{ij} can be positive, negative, or zero. $\gamma_{i,n+1} \equiv 0$.

THEOREM 10: For the cost operator δC_A, the optimal dual solutions are given by

$$\begin{aligned} u_i^A &= u_i + \delta u_i^* \qquad \text{for } i \epsilon I' \\ v_j^A &= v_j + \delta v_j^* \qquad \text{for } j \epsilon J'. \end{aligned} \tag{46}$$

The maximum extent to which this basis preserving operator can be applied is given by:

$$\mu^A = \text{Min} \begin{cases} (c_{ij} - e_{ij}u_i - v_j)/(e_{ij}u_i^* + v_j^* - \gamma_{ij}) & \text{for } (i,j) \epsilon \{LB \mid (\gamma_{ij} - e_{ij}u_i^* - v_j^*) < 0\} \\ (e_{ij}u_i + v_j - c_{ij})/(\gamma_{ij} - e_{ij}u_i^* - v_j^*) & \text{for } (i,j) \epsilon \{UB \mid (\gamma_{ij} - e_{ij}u_i^* - v_j^*) > 0\}. \end{cases} \tag{47}$$

μ^A is set equal to M, if the sets over which the minimization was done turn out to be empty.

PROOF: First we show that the dual solutions (46) satisfy (17) for the problem P^A. From (45) and (46) we get $e_{ij}u_i^A + v_j^A = (e_{ij}u_i + v_j) + \delta(e_{ij}u_i^* + v^*) = c_{ij} + \delta\gamma_{ij} = c_{rj}^A$. For $(i, j) \epsilon LB$, $c_{ij}^A - e_{ij}u^A - v_j^A = (c_{ij} - e_{ij}u_i - v_j) + \delta(\gamma_{ij} - e_{ij}u_i^* - v_j^*)$. Since $c_{ij} - e_{ij}u_i - v_j \geq 0$ for the problem P for $0 \leq \delta \leq \mu^A$ as defined by (47), the dual feasibility condition (18) is satisfied by these cells. For $(i, j) \epsilon UB$, $e_{ij}u^A + v_j^A - c_{ij}^A = (e_{ij}u_i + v_j - c_{ij}) + \delta(e_{ij}u_i^* + v_j^* - \gamma_{ij})$. Since $e_{ij}u_i + v_j - c_{ij} \geq 0$ for problem P, for $0 \leq \delta \leq \mu^A$ as defined by (47) the dual feasibility condition (19) is satisfied by these cells. Thus the solution (46) is dual feasible. Since $X^A = X$ is primal feasible as well, it is optimal. The quantity μ^A gives the maximum δ, since for $\delta > \mu^A$, the *entering cell* (e_1, f_1) at which the minimum of (47) is attained will violate dual feasibility.

The results of this section are summarized in the form of Algorithm A10 for determining μ and Algorithm A11 for applying operator δC_A.

ALGORITHM A10: For determining μ associated with basis-preserving cost operators and the entering cell (e_1, f_1)

(1) If μ_A is to be determined (δC_A) go to (4). If $\mu^{\pm}$ is to be determined $(\delta C^{\pm})$ go to (2).

(2) If $(p, q) \epsilon B$ go to (3). Otherwise determine $\mu^+(\mu^-)$ by using (36) ((37)). Record $(e_1, f_1) = (p, q)$. STOP.

(3) Find the sets I_p, J'_p, I_q and J'_q using Definition 17 and Remark 14. Determine μ^+, (μ^-) by using (40), (41), ((43), (44)). Record (e_1, f_1) as the cell at which the minimum is taken on. STOP.

(4) Find u_i^*, v_j^* satisfying (45). Determine μ^A using (47). Record (e_1, f_1) as the cell at which the minimum is taken on. STOP.

ALGORITHM A11: For applying the basis preserving cost operator.

(1) If it is an area operator (δC_A) go to (4). If it is a cell operator $(\delta C_{pq}^{\pm})$ go to (2).

(2) Set the data of problem $P^{\pm}$ same as problem P except that $c^{\pm}_{pq}=c_{pq}\pm\delta$. Set $B^{+}=B$, $LB^{\pm}=LB$; $UB^{\pm}=UB$, $X^{\pm}=X$ and $Z^{\pm}=Z\pm\delta x_{pq}$. If $(p, q)\epsilon B$. Go to (3). Otherwise set $D\pm=D$. STOP.

(3) Find the sets I_p, J'_p, I_q and J'_q by using Definition 17 and Remark 14. Determine $D^{+}(D^{-})$ by using (39) ((42)). STOP.

(4) Set the data of P^{A} same as P except that $c^{A}_{ij}=c_{ij}+\delta\gamma_{ij}$ for $i\epsilon I'$ and $j\epsilon J'$. Set $B^{A}=B$, $LB^{A}=LB$, $UB^{A}=UB$, $X^{A}=X$ and $Z^{A}=Z+\delta\left[\sum_{i\epsilon I'}\sum_{j\epsilon J'}\gamma_{ij}x_{ij}\right]$. Find u_i^{*} and v_j^{*} satisfying (45). Determine D^{A} using (46). STOP. (D represents the dual matrix [24].)

EXAMPLE (continued): Referring to Fig. 4, [24], for the operator δC^{+}_{12} from (36) we get $\mu^{+}=M$, i.e., the solution of Fig. 4 is optimal regardless of how large c_{12} becomes. For the operator δC^{-}_{12}, however, $\mu^{-}=6-0-28/5=2/5$ from (37). Similarly $\mu^{+}=31/5-3=16/15$ for δC^{+}_{11}. For all these operators the optimum solutions X and D remain the same.

Consider now the operator δC^{+}_{21}. (2, 1) ϵB and ϵB_2; Utilizing Definition 17 and Equation (38) $I_p=\{2, 3\}$; $J'_{pk}=\{\emptyset\}$; $I_{qk}=\{\emptyset\}$; $J'_q=\{1, 2\}$, $J'_p=\{3, 4\}$; $I_q=\{1, 4\}$. Let us apply Algorithm A9(a). Since $(2, 1)\epsilon C_2\subseteq T_2$.

Steps of A9(a)	*Result due to application of this step.*
(0)	$C=\{(2, 1), (2, 2), (3, 2), (3, 1)\}$; $C^{*}=\emptyset$. Let $(i, j)=(2, 1)$. Since $p(C_1)=R_2$ set $m_2=(1-\rho)/e_{ij}$ $=-1/5$ and $u_2=u_2+m_2\delta=-1/5-1/5\delta$. Go to (2).
(2)	$C_2=p(R_2)$. $n_2=-e_{22}m_2=3/5$; $v_2=v_2+n_2\delta$ $=28/5+3\delta/5$. Go to (1).
(1)	$R_3=p(C_2)$; $m_3=-n_2/e_{32}=-3/10$; $u_3=u_3+m_3\delta$ $=-13/10-3\delta/10$.
(2)	$C_1=p(R_3)$; $n_1=-e_{31}m_3=6/5$; $v_1=v_1+n_1$ $=31/5+6/5\delta$. Since $C_1=f$ go to (3).
(3)	$p(C_1)=\emptyset$. STOP. Thus we have $u_2^{+}=u_2-\delta/5$, $u_3^{+}=u_3-3\delta/10$ $\}$ $i\epsilon I_p$; and $v_1^{+}=v_1+6\delta/5$, $v_2^{+}=v_2+3\delta/5$ $\}$ $i\epsilon J'_q$; $u_i^{+}=u_i$ for $i\epsilon I-I_p(=I_q)$; $v_j^{+}=v_j$ for $j\epsilon J'-J'_q(=J'_p)$.

These dual solutions are given as Figure 11 for general δ and Figure 12 for $\delta=2/3$. Cost increases at the rate of $x_{21}=135$, since none of the other costs are affected. (Theorem 7(a)). We find $\mu^{+}=2/3$ limited by the cell $(e_1, f_1)=(1, 2)$. The reader may verify that the dual solutions of Figures 11 and 12 are dual feasible (i.e., satisfy (18) and (19)). Since the primal solution remains the same it is optimal as well.

$\delta C_{21}^{+}(P)$

u_i \ v_j	$\frac{31}{3}+\frac{6\delta}{5}$	$\frac{28}{5}+\frac{3\delta}{5}$	3	4	–	a_i
0	3 [3] 15 [15]	6 [2] [N]	(3) [2] 35 [40]	(4) [3] 60 [80]	(0) [1] 5 [N]	300
$-1/5$ $-\delta/5$	(6+δ) [1] 135 [150]	(5) [3] 30 [75]	8 [5] [25]	15 [10] [N]	0 [1] [N]	225
$-13/10$ $-3\delta/10$	(1) [4] 20 [60]	(3) [2] 30 [N]	10 [2] [30]	6 [1] 50	0 [1] [N]	140
0	100 [1] [N]	100 [1] [N]	100 [1] [N]	100 [1] [N]	(0) [1] [N]	1000
b_j	170	60	35	60	–	

FIGURE 11.

$(2/3)C_{21}^{+}(P)$

u_i \ v_j	7	6	3	4	–	a_i
0	3 [3] 15 [15]	6 [2] [N]	(3) [2] 35 [40]	(4) [3] 60 [80]	(0) [1] 5 [N]	300
$-1/3$	(6+2/3) [1] 135 [150]	(5) [3] 30 [75]	8 [5] [25]	15 [10] [N]	0 [1] [N]	225
$-3/2$	(1) [4] 20 [60]	(3) [2] 30 [N]	10 [2] [30]	6 [1] 50	0 [1] [N]	140
0	100 [1] [N]	100 [1] [N]	100 [1] [N]	100 [1] [N]	(0) 1000 [1] [N]	1000
b_j	170	60	35	60	–	

FIGURE 12.

We now consider δC_A with γ_{ij} as shown in Figure 13. The values of u_i^* and v_j^* satisfying (43) are displayed on the left and top rims of Figure 13. By using (45) $\mu_A = 1/3$ and the minimum is attained at the cell $(e_1, f_1) = (2, 5)$. Application of $1/3\ C_A$ using Algorithm A11 results in the tableau of Figure 14. Again, the solution may be verified to be dual feasible and hence optimal. From Theorem 7(c), $Z^A = 1{,}460 + 545 = 2{,}005$ which agrees with $\sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij}$ directly computed from Figure 14.

In this problem all the μ's turn out to be strictly positive. It is easy to infer from (39), (40), (43), (45), and (48) that this need not be the case if the solution is dual degenerate (i.e., corresponding to a nonbasic cell $c_{ij} = e_{ij} u_i + v_j$).

$(\gamma_{ij}, u_i^*, v_j^*)$
CIRCLED CELLS ARE BASIC CELLS
UB: CELLS ARE MARKED
LB; BLANK CELLS ARE LB CELLS
γ_{ij} ARE GIVEN IN THE CENTER; θ_{ij} IN THE NORTH WEST CORNER

u_i^* \ v_j^*	2/5	1/5	2	3	–
0	3 UB −2	2 +1	2 (2)	3 (3)	1 (0)
3/5	1 (1)	3 (2)	5 7	10 10	1 0
2/5	4 (2)	2 (1)	2 3	1 4	1 0
0	1 0	1 0	1 0	1 0	1 (0)

FIGURE 13.

$1/3\ C_A(P)$
$u_i^A = u_i + \delta u_i^*;\quad v_j^A = v_j + \delta v_j^*;\quad c_{ij}^A = c_{ij} + \delta\gamma_{ij}$

u_j^A \ v_j^A	19/3	17/3	11/3	5	–	a_i
0	3 (2-1/3) 15 / 15	2 6-1/3 N	2 (3-2/3) 35 / 40	3 (5) 60 / 80	1 (0) 5 / N	300
0	1 (6-1/3) 135 / 100	3 (5-2/3) 30 / 75	5 9-1/3 / 25	10 18-1/3 / N	1 0 / N	225
−7/6	4 (1-2/3) 20 / 60	2 (3-1/3) 30 / N	2 11 / 30	1 7-1/3 / 50	1 0 / N	140
0	1 100 / N	1 100 / N	1 100 / N	1 100 / N	1 (0) 1000 / N	1000
b_j	170	60	35	60	–	

FIGURE 14.

4. BOUND OPERATORS

In this section we show that bound operators which arise when upper bounds alone are changed, can be reduced to application of rim operators. For the case of lower bounds, propositions 1 and 2 are applicable. We also show that simultaneous changes of costs and rims conditions can be effected by sequentially applying rim and cost operators. The discussion below is restricted to basis preserving operators.

DEFINITION 18: A *bound operator* $\delta L(P)$ transforms the optimum solution of problem P into

those of a transformed problem P^+ whose data are the same as those of P except that

(a) $U^+_{pq} = U_{pq} + \delta$ for the δL^+_{pq} operator where (p, q) is given index pair.

(b) $U^-_{pq} = U_{pq} - \delta$ for the δL^-_{pq} operator where (p, q) is a given index pair, and

(c) $U^A_{ij} = U_{ij} + \delta\nu_{ij}$ for $i \epsilon I'$ and $j \epsilon J'$ for δL_A operator.

For $\delta L(P)$ to be basis preserving, by definition 15, we have $B^{\pm}(B^A) = B$, $LB^{\pm}(LB^A) = LB$ and $UB^{\pm}(UB^A) = UB$. The last of these equations has to be interpreted as follows: the cells that are nonbasic at their upper bonds of P still remain nonbasic at their (possibly different) upper bounds for the transformed problem.

THEOREM 11: (a) For the operator δL^+_{pq} with $(p, q) \notin UB$ (48)

$$X^+ = X;\ D^+ = D;\ Z^+ = Z \text{ and } \mu^+ = M$$

(a large positive quantity)

(b) For the operator δL^-_{pq} with $(p, q) \notin UB$ (49)

$$X^- = X;\ D^- = D;\ Z^- = Z \text{ and } \mu^- = U_{pq} - x_{pq}.$$

PROOF: Since $(p, q) \notin UB$ changing U_{pq} to $U_{pq} \pm \delta$ affects neither the primal constraints (6), (7) nor the dual feasibility conditions (17)–(19). The choice of $\mu^{\pm}$ in (48), (49) is such that for $0 \leq \delta \leq \mu^{\pm}$ the primal feasibility condition (8) is also satisfied so that the solution is optimal. Thus $X^{\pm} = X$ and $D^{\pm} = D$. As x_{ij} and C_{ij} have not changed $Z^{\pm} = Z$.

Let us now consider δL^+_{pq} with $(p, q) \epsilon UB$. In order that this be basis-preserving we should set $x^+_{pq} = x^+_{pq} + \delta = U_{pq} + \delta = U^+_{pq}$. But increasing x_{pq} and keeping the other x_{ij}'s unaltered violates (6) for $i = p$ and (7) for $j = q$, with e_{pq} being unchanged. To counteract this we can apply the rim operator δR^-_{pq} and modify the x_{ij}'s of basic cells so that (6) and (7) again hold. This can be done to a maximum extent of $\mu(R^-_{pq})$ (i.e., the μ^- corresponding to the operator δR^-_{pq}). Since x_{pq} increased the resulting increase in the rate of cost is c_{pq}, whereas the δR^-_{pq} operator yields a decrease in total cost at the rate of d_{pq} Theorem 3 (b)), so that the net cost decrease is at the rate of $d_{pq} - c_{pq} = w_{pq}$ as w_{pq} is given to be $u_p e_{pq} + v_q - c_{pq}$. Similar results apply to the δL^-_{pq} operator with $(p, q) \epsilon UB$. However, since U_{pq} cannot be decreased below zero we have $\delta \leq U_{pq}$. These results are summarized in the following theorem.

THEOREM 12: (a) For the operator δL^+_{pq} with $(p, q) \epsilon UB$, $Z^+ = Z - \delta w_{pq}$; $D^+ = D$; X^+ is obtained by applying the operator δR^-_{pq} to P and then setting $x^+_{pq} = x_{pq} + \delta$; $\mu(L^+_{pq}) = \mu(R^-_{pq})$.

(b) For the operator δL^-_{pq} with $(p, q) \epsilon UB$, $Z^- = Z + \delta w_{pq}$ and $D^- = D$; X^- is obtained by applying δR^+_{pq} to P and then setting $x^-_{pq} = x_{pq} - \delta$; $\mu(L^-_{pq}) = \text{Min}\,(U_{pq}, \mu(R^+_{pq}))$.

The area operator δL_A can also be applied likewise. We need to set $x^A_{ij} = x_{ij} + \delta v_{ij}$ for $(i, j) \epsilon UB$, to enable the basis preserving property. To counteract the effect of these changes on (6) and (7) we will have to apply δR_A with $\alpha_i = -\sum_{\{j \epsilon J \mid (i,j) \epsilon UB\}} e_{ij}\nu_{ij}$ for $i \epsilon I$ and $\beta_j = -\sum_{\{i \epsilon I \mid (i,j) \epsilon UB\}} \nu_{ij}$ for $j \epsilon J$. The new optimal cost $Z^A = Z - \delta \sum_{i \epsilon I} \sum_{j \epsilon J} \nu_{ij} w_{ij}$ (since by definition of w_{ij} (20), $w_{ij} = 0$ for $(i, j) \epsilon UB$).

Let us consider the case of rim and cost operators simultaneously applied, i.e., suppose that the changes in the data are as given by (22) with $\nu_{ij} = \epsilon_{ij} = 0$ for $i \epsilon I'$; $j \epsilon J'$.

It is known (section 2) that the application of rim operators affect only dual variables so that there is no interaction effect between these operators. In other words, $\delta T(P)$ can be applied by first applying the associated $\delta R_A(P)$ and then applying the associated $\delta C_A(P)$ on the transformed problem. This operator is denoted as $\delta C_A(\delta R_A(P))$ which can be easily seen to be equal to $\delta R_A(\delta C_A(P))$. Since the final solution should be both primal and dual feasible it follows that

$$\mu(T(P)) = \text{Min}[\mu(R_A(P)), \mu(C_A(P))].$$

The cost effect will be the sum of the two effects due to rim and cost operators [i.e., $Z^T = Z + \delta\left[\sum_{i\epsilon I} \alpha_i u_i + \sum_{j\epsilon J} \beta_j v_j + \sum_{i\epsilon I}\sum_{j\epsilon J} \gamma_{ij} x_{ij}\right]$. Such results can be naturally extended to the simultaneous application of $\delta R^{\pm}_{pq}$ with δC_A or that of δR_A with $\delta C^{\pm}_{pq}$ etc.

It can be noted that a simultaneous application of cell cost operators say δC^{+}_{ij} for $(i, j)\,\epsilon\Psi$ and δC^{-}_{ij} for $(i, j)\,\epsilon\chi$ (Ψ and χ are non-lapping subsets of $I \times J$) is equivalent to an area operator δC_A with $\gamma_{ij} = 1$ for $(i, j)\,\epsilon\Psi$ and $= -1$ for $(i, j)\,\epsilon\chi$ with $\gamma_{ij} = 0$ elsewhere. Similar statements can be made for cell rim operators as well.

5/3 L^{+}_{11} (P)

u_i \ v_j	31/5	28/5	3	4	–	a_i
0	3 15+5/3 3 15+5/3	2 6 N	2 35 3 40	3 60 4 80	1 0 0 N	300
-1/5	1 133 6 150	3 302/3 5 75	5 8 25	10 15 N	1 0 N	225
-13/10	4 20-1/3 1 60	2 29-1/3 3 N	2 10 30	1 6 50	1 0 N	140
0	1 100 N	1 100 N	1 100 N	1 100 N	1 1000 0 N	1000
b_j	170	60	35	60	–	

FIGURE 15.

EXAMPLE (continued): Figure 15 illustrates the result of applying bound operator L^{+}_{11} to problem P of Figure (4) [24]. Since $(1, 1)\,\epsilon UB$, the application of this operator involves increasing x_{11} to $x_{11} + \delta$ and applying $\delta R_{\bar{1}1}$ using Algorithm (8).

Referring to Figure 4 [23] the two-tree Γ created by adding (1, 1) to the basis is,

$$\Gamma = \{[(1, 5)] \cup [(1, 1)] \cup [(2, 1), (2, 2), (3, 2), (3, 1)]\}$$

$$(C_h \cup D_u) \cup (D_t \cup C_t)$$

with $\Gamma_1 = \{(1, 1), (2, 2), (3, 1)\}$, 'giver' cells and
$\Gamma_2 = \{(1, 5), (2, 1), (3, 2)\}$, 'getter' cells.

From (31) $\mu(R_{\bar{1}1}) = \text{Min}\left(\frac{x_{15}}{|m_{15}|}, \frac{x_{21}}{|m_{21}|}, \frac{x_{32}}{|m_{32}|}, \frac{U_{31} - x_{31}}{m_{31}}\right)$

$$= \frac{x_{15}}{|m_{15}|} = \frac{5}{3} = \mu(L^{+}_{11}).$$

Applying 5/3 R_1^{-1} and changing x_{11} from 15 to 15 5/3 results in the tableau of Fig. 15 which can be seen to be optimal as it is both primal and dual feasible. Since $w_{11} = e_{11}u_1 + v_1 - c_{11} = 16/5$, by Theorem 12(a) $Z^+ = Z - \delta w_{11} = 1460 - 5/3 \times 16/5 = 1454\ 2/3$ which agrees with $\sum_{i \in I} \sum_{j \in J} c_{ij}x_{ij}$ computed directly from Fig. 15.

1/3 R_A [1/3 C_A (P)] = 1/3 C_A [1/3 R_A (P)]

u_i \ v_j →	19/3	17/3	11/3	5	–	a_i
0	3 · 15 · 2-1/3 · 15	2 · 6-1/3 · N	2 · 35-2/3 · 3-2/3 · 40	3 · 57-2/3 · (5) · 80	1 · 0 · (0) · N	299-1/3
0	1 · 137-1/15 · (6-1/3) · 150	3 · 29-8/15 · (5-2/3) · 75	5 · 9-1/3 · 25	10 · 18-1/8 · N	1 · 0 · N	225-2/3
−7/6	4 · 19-3/5 · (1-2/3) · 60	2 · 31-7/15 · (3-1/3) · N	2 · 11 · 30	1 · 7-1/3 · 50	1 · 0 · N	141-1/3
0	1 · 0 · 100 · N	1 · 0 · 100 · N	1 · 0 · 100 · N	1 · 0 · 100 · N	1 · 1000 · (0) · N	1000
b_j	171-2/3	61	35-2/3	57-2/3	–	

FIGURE 16.

Figure 16 illustrates the simultaneous application of the area rim operator of Figure 13. As shown earlier $\mu(R_A) = 75/31$ and $\mu(C_A) = 1/3$ so that they can be applied to a maximum extent of $\delta = 1/3$ resulting in the Tableau of Figure 16. This is optimal as this solution is both primal and dual feasible which can be verified.

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THE KNAPSACK PROBLEM: A SURVEY*

Harvey M. Salkin

Case Western Reserve University

Cornelis A. de Kluyver

Purdue University

ABSTRACT

A unifying survey of the literature related to the knapsack problem; that is, maximize $\sum_i v_i x_i$, subject to $\sum_i w_i x_i \leqslant W$, $x_i \geqslant 0$, and x_i integer; where v_i, w_i, and W are known integers, and $w_i (i = 1, 2, \ldots, N)$ and W are positive. Various uses, including those in group theory and in other integer programming algorithms, as well as applications from the literature, are discussed. Dynamic programming, branch and bound, search enumeration, heuristic methods, and other solution techniques are presented. Computational experience, and extensions of the knapsack problem, such as to the multi-dimensional case, are also considered.

1. INTRODUCTION

This survey considers the knapsack problem: maximize $\sum_{i=1}^{N} v_i x_i$, subject to $\sum_{i=1}^{N} w_i x_i \leqslant W$, $x_i \geqslant 0$, and x_i integer (or x_i binary), for $i = 1, 2, \ldots, N$; where v_i, w_i and W are known integers and w_i $(i = 1, 2, \ldots, N)$ and W are positive. As this integer program may represent a large number of situations it is extensively discussed in the literature. In words, the problem is, given the weight and desirability (i.e., w_i and v_i, respectively) of each of N items, to find that subset of the items that maximizes total desirability and satisfies a constraint on total weight (W). In almost all applications the costs v_i $(i = 1, 2, \ldots, N)$ are positive. A distinct case where this may not occur is when an integer program is transformed to an equivalent knapsack problem (section 3).

The article is organized in several parts. First, some applications and uses of the knapsack problem are reviewed. Secondly, some recent results in transforming integer linear programs to the knapsack problem are surveyed. Thirdly, algorithms are discussed in four sections: dynamic programming, integer programming, heuristics and lagrangian methods, and network approaches. Although this classification is sometimes arbitrary, it gives the reader an indication of the directions taken. Finally, a section on computational experience is included. Comparisons are often difficult because of different machines, computer codes, methods, and test problems used. Nevertheless, the numerical results should give the reader some indication of efficiency.

2. APPLICATIONS AND USES

Although the knapsack problem may represent many industrial situations such as cargo loading, project selection, budget control, etc., relatively few applications are commented on in the literature.

Lorie and Savage [60] describe applications in capital budgeting. The multi-period Lorie-Savage problem concerns itself with choosing among competitive investment possibilities so as to maximize total payoff over all periods. The payoffs and cash requirements for one project are assumed to be independent of another investment possibility. The constraints limit total cash outlay in each period. The total payoff over all periods is assumed to be a linear combination of the payoffs of the individual investments which is the case when measures such as present value are used, and when the investments are unrelated. The multi-period model has as many constraints as periods and thus, excpet for the one-period model, is not a knapsack problem. However, as the constraint coefficient matrix is nonnegative and the right-hand side is positive it is often discussed in the context of a multi-dimensional knapsack problem. Lagrangian multiplier techniques applied to this problem are discussed by Cord [21] and Kaplan [50].

Weingartner [76, 77, 78] discusses the Lorie-Savage problem extensively and extends the model to the case where the projects are *not* independent. In addition, Weingartner discusses the relative merits of integer programming methods to solve these problems. Extensions of the model, not all of the knapsack type, have also been considered. In particular, the case where the objective function is quadratic is treated by Unger [74], Radhakrishan [67], and by Mao and Wallingford [61]. The use of Balas' duality theory [3, 4, 5] in integer programming to give economic interpretations and properties of optimal solutions (for the linear and nonlinear model) is discussed by Radhakrishan [67]. Other articles relating mathematical programming to capital budgeting include those by Baumol and Quandt [6], Byrne, Charnes, Cooper, and Kortanek [18], Naslund [64], and Woolsey [79].

A recent paper by Glover and Klingman [36] presents an application to the problem of selecting journals in a library. The model seeks to determine the optimal allocation of present and future (expected) funds to purchase scientific journals using an objective function on expected usage as a measure of journal worth. Earlier articles describing the problem include those by Kraft and Hill [53, 54].

Related to the knapsack problem is the cutting-stock problem as described by Gilmore and Gomory [29, 30, 31]. The cutting-stock problem is described by a linear program where a column represents a certain cutting pattern. As there are a vast number of possible cutting patterns, a column generation scheme is used. It is in this context that the knapsack problem arises. The specific nature of this problem will be discussed in more detail in the algorithm section of this article.

A recent application of the knapsack problem is in the use of group theory in integer programming. In particular, Gomory and Johnson [40] have shown that, by dropping feasibility requirements on certain variables, any integer program can be defined as an integer program on an abelian group. Moreover, the algorithms (see e.g., Hu [49]) for this problem, which are usually of the dynamic programming type, treat the group relationships as a single constraint and thus are essentially tailored to solve a knapsack problem.

3. REDUCING INTEGER LINEAR PROGRAMS TO KNAPSACK PROBLEMS

Mathews [62] has shown that two equations whose coefficients are all positive integers can be aggregated into a single equation with the same solution set when the variables are restricted to nonnegative integers. The theorem of Mathews [62] may be stated as follows:

THEOREM: Consider a system of two equations with strictly positive integer coefficients

$$s_1 = \sum a_{1j} x_j = b_1 \tag{1}$$

2) $$s_2 = \sum_j a_{2j} x_j = b_2.$$

Then: (a) If there exist nonnegative integer values for the variables x_j satisfying (1) and (2), then $b_2 (a_{ij}/a_{2j}) \geq b_1$ for at least one j.

(b) If t is any positive integer such that

$$t > b_2 \max_j (a_{1j}/a_{2j}) \geq b_1,$$

then the solution set of (1) and (2) in nonnegative integer variables is the same as that of the single equation

(3) $$s_1 + ts_2 = b_1 + tb_2.$$

Mathews' theorem allows the reduction of a multi-constraint integer programming problem with a nonnegative (but not all positive) integer coefficient matrix to a single-constraint integer program for which knapsack algorithms are available. Specifically, the theorem can be applied to two equations (1) and (2) with nonnegative inter coefficients by first replacing them with

(4) $$s_1 + s_2 = b_1 + b_2$$

(5) $$s_1 + 2s_2 = b_1 + 2b_2.$$

These equations have the same integer solutions as (1) and (2) and have positive integer coefficients (since columns of all zero's may be dropped) and hence can take the role of (1) and (2) in Mathews' theorem.

Elmaghraby and Wig [25] extended these results for transforming inequality constraints into a single equation. The method consists of pairwise aggregating equations of the form (1) and (2) by using the equivalent equations (4) and (5). Negative coefficients are treated by complementing variables.

Glover and Woolsey [33] have shown that the applications of (4) and (5) are severely limited due to a greater than exponential growth in the size of the coefficients as successive constraints are aggregated into one. These authors give several more efficient ways of utilizing Mathews' results which do not utilize the transformation of (4) and (5). In particular, if we denote the system of equations (after adding slacks y_i) by

(6) $$s_i = \sum_j a_{ij} x_j + y_i = b_i \qquad (i = 1, 2, \ldots, m)$$

the results are implemented by replacing equation i in (6) by the sum of the first i equations, and a new aggregation scheme is applied.

In subsequent sections of their article, Glover and Woolsey [33] generalize Mathews' theorem (i.e., weaken the assumptions) to improve on the aggregation process by allowing smaller values of t. In their concluding remarks, the authors [33] consider the possibilities for reverse procedures, that is, to

disaggregate an aggregated equation into component equations. The authors point out that this approach may be useful as part of a cutting plane strategy. (Since new equations or cuts appear.)

More recent papers on the subject of reducing integer linear programs to knapsack problems include those of Anthonisse [1], Bradley [13, 14], Garfinkel and Nemhauser [28], Glover [35], Hammer and Rudeanu [48], and Padberg [66], generalizing the results of Mathews and proposing better weights to prevent the magnitude of the coefficients from growing too rapidly. An excellent survey is given by Kendall and Zionts [51].

It should be pointed out that the potential of these methods crucially depends on our ability to solve the general single-constraint knapsack problem which may have negative costs and bounded variables.

Algorithms are discussed next.

4. ALGORITHMS

A variety of techniques have been developed in recent years to solve the kanpsack problem. For ease of exposition, we have classified these methods in the following categories:

(i) dynamic programming or variants thereof
(ii) integer programming methods
  branch and bound
  other enumerative algorithms
(iii) heuristic search and lagrangian methods
(iv) network approaches.

This classification is sometimes arbitrary as algorithms may have features of several categories. It should, however, give the reader an indication of the approach taken.

4.1 Dynamic Programming Algorithms and Variants

The most common technique for solving knapsack problems is dynamic programming (DP). Early papers which discuss the knapsack problem in the context of dynamic programming include those by Bellman [7, 8, 9, 10], and Dantzig [22]. Also, Bellman and Dreyfus [11], Wagner [75], Weingartner and Ness [76] discuss applications involving relatively few variables. Refinements of the DP-approach have been developed by Gomory [39], Gomory and Gilmore [29, 30, 31, 32], Greenberg [41], Dreyfus and Prather [24], and Yormark [80]. These refinements mostly consist of "look-ahead" methods, the development of efficient bounds, and exploitation of specific data-structures.

The general DP approach, attributed to Bellman [9], is based on what items would be included in the solution if:

(i) the weight limitation were $w^0 = 1, 2, \ldots, W$, and
(ii) the selections were restricted to only the first k items $(k=1, 2, \ldots, N)$.

For example, if $k = 1$ and $w^0 < w_1$ (where w_1 is the weight of the first item) then the item would not be selected, but it would be if $w^0 \geq w_1$. This approach is repeated inductively for $w^0 = 1, 2, \ldots, W$ and for $k=1, 2, \ldots, N$.

Define $F_k(w^0)$ to be the maximal value of the objective function using only the first k $(k=1, 2, \ldots, N)$ items with weight limitation w^0 $(w^0=1, 2, \ldots, W)$. That is, $F_k(w^0) = \max \sum_{i=1}^{k} v_i x_i$, such that

$\sum_{i=1}^{k} w_i x_i \leq w^0$, $x_i \geq 0$ and integer $(i = 1, 2, \ldots, k)$. It is easily shown (see e.g., Garfinkel and Nemhauser [28], or Salkin [68]) that

$$F_k(w^0) = \text{Maximum } \{v_k x_k + F_{k-1}(w^0 - w_k x_k)\}, \tag{7}$$

$$x_k = 0, 1, \ldots, \left[\frac{w^0}{w_k}\right]$$

where $[y]$ means the largest integer $\leq y$. Equation (7) can be used recursively to find $F_k(w^0)$ for $k = 1, 2, \ldots, N$ and $w_0 = 1, 2, \ldots, W$. This approach is especially useful when the variables are bounded (in which case u_k replaces $\left[\frac{w^0}{w_k}\right]$ in (7) whenever $u_k < \left[\frac{w^0}{w_k}\right]$, where u_k is the integer upper bound for x_k, $k = 1, 2, \ldots, N$) and can easily be generalized to handle a nonlinear objective function and (or) constraint.

Using the linearity of the objective function and constraint a computationally more efficient recursion can be developed. As the $F_k(w^o)$ problem differs from the $F_{k-1}(w^o)$ problem in only one variable, namely x_k, it must be decided whether x_k should be fixed at zero or be at least one.

Clearly, (i) $x_k = 0$ if $F_k(w^0) \leq F_{k-1}(w^0)$

(ii) $x_k \geq 1$ if $v_k + F_k(w^o - w_k) > F_{k-1}(w^o)$ and $w^o - w_k \geq 0$.

Thus,

$$F_k(w^o) = \max_{k=1,2,\ldots,N} \{F_{k-1}(w^o), v_k + F_k(w^o - w_k)\} \text{ for } w_k \leq w^o. \tag{8}$$

The new recursion (8) can then be solved inductively for $i = 1, 2, \ldots, N$ and $w^o = 1, 2, \ldots, W$. The reader can verify that this procedure satisfies the principle of optimality and results in a smaller number of computations. By defining $F_o(0) = 0$ and $F_o(w^o) = -\infty$ for $w^o > 0$, recursions (7) or (8) solve the knapsack problem with an equality constraint. Even higher efficiency is obtained when only "breakpoints" in the weights are considered. That is, instead of discretizing the problem as described above where w^0 successively takes on the values $w^0 = 1, 2, \ldots, W$, we only consider the points $w^0 = w_1$, $w^0 = w_2$, $w^0 = w_3$, $w^0 = w_1 + w_2$, etc., after the w_i have been reordered such that $w_i \leq w_{i+1}$, $i = 1, 2, \ldots, N$. Thus, the only values for w^0 that are evaluated are those which are certain linear combinations of the w_i such that $w^0 \leq W$. In addition, the algorithm only keeps track of the last optimal selection thus minimizing computer storage requirements.

Weingartner and Ness [76] point out that, when the variables are required to be zero or one, it may be easier to solve what they define as "the complement problem." That is, we start with an infeasible solution with all variables at their upper bound, and then systematically delete items from this full loading. At the last step, feasibility and optimality are reached simultaneously. This approach is easily defined in a dynamic programming relation from the previous recursion by a simple transformation of variables (see [56]). The paper contains additional refinements including the use of lower bounds, a discussion of upper bounds, and a specific interactive computer code with computational results.

An excellent treatment of knapsack problems and dynamic programming algorithms can be found in Gilmore and Gomory [29, 30, 31, 32]. The discussion describes applications in the paper industry

in context of cutting-stock problems. The cutting-stock problem (in the notation of [17]) is defined as:

$$\text{Minimize } \sum_j x_j, \qquad \text{subject to } \sum_j a_{ij}x_j \geq N_i, \qquad \text{where } N_i, \qquad i = 1, 2, \ldots, m$$

denotes the number of rolls of length l_i demanded, x_j is the number of times the jth cutting pattern is used and a_{ij} is the number of rolls of length l_i produced each time the jth cutting pattern is used. The usually huge number of cutting patterns (columns) constitutes a major difficulty in solving this problem. This difficulty is circumvented by employing a column generation scheme, where each column to enter the basis is generated by solving a knapsack problem. In particular, in order to produce the next column, we have to find that column whose scalar product with the current linear programming dual evaluators is maximal. Since any set of nonnegative integers a_i $(i = 1, 2, \ldots, m)$ is a column (or cutting pattern) provided that $\sum_{i=1}^{m} l_i a_i \leq L$, where L is the standard stock length, a column is generated from the problem:

$$\text{Maximize } \sum_{i=1}^{m} b_i a_i, \qquad \text{subject to } \sum_{i=1}^{m} l_i a_i \leq L, \qquad \text{and } a_i \text{ a nonnegative integer for } i = 1, 2, \ldots, m.$$

Here the b_i denote the current linear programming dual evaluators (note that this procedure is possible since all costs of the x_j are equal in the linear programming formulation for all j). Hence, a knapsack problem is solved repeatedly, until we cannot find an optimal value which exceeds the cost C of the standard stock length L. At this point, the cutting stock problem is solved because no marginal profit can be obtained from any other cutting problem. The Gilmore and Gomory approach requires an efficient algorithm to solve the knapsack problem. The recursive methods described previously may be employed. Other algorithms, developed by the same authors, are in [19]. These techniques are based on the theory of knapsack functions. Loosely, a one-dimensional knapsack function is one which maximizes a linear objective function subject to a single constraint in nonnegative variables. Necessary and sufficient conditions for a function to be a knapsack function are developed. The article also discusses periodic properties of one-dimensional knapsack functions, algorithms and computational results. The algorithms are of the dynamic programming type, using a "forward" recursion and considering only the breakpoints in the weights.

In a later paper, Shapiro and Wagner [69] establish the mathematical and algorithmic connections between the knapsack problem and a fundamental renewal-type problem. Although the algorithm employs dynamic programming, the network formulation as a shortest route problem is the dominant feature of the approach and it will therefore be discussed in a subsequent section on network approaches.

In their paper on the journal selection problem for libraries, Glover and Klingman [36] discuss approximate methods to solve a highly structured zero-one integer program (Figure 1). The first method, proposed by Kraft and Hill [53, 54] involves examining each budget constraint (Figure 1) in succession as if it is the only constraint in the problem. This process is repeated for each of these constraints and an optimal solution is obtained for each of these one-dimensional subproblems (e.g., by zero-one dynamic programming). Once an optimal solution for a subproblem is obtained, this solution is enforced in the next subproblem by appropriately reducing the right-hand sides of the budget constraints as a

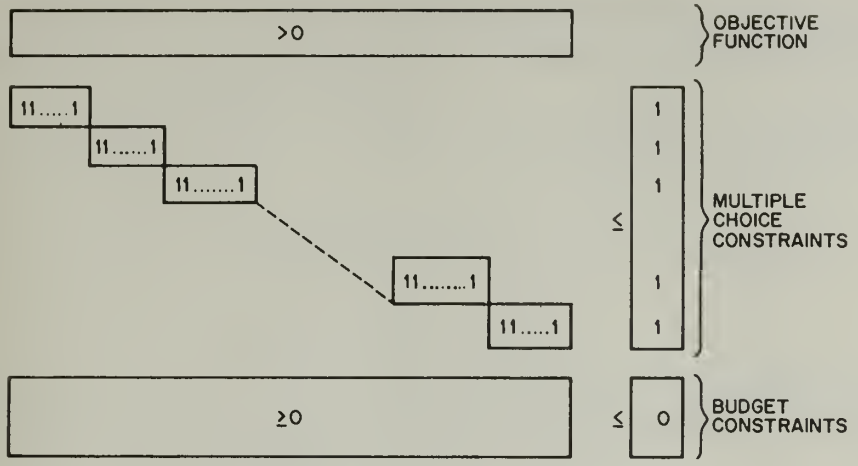


FIGURE 1. Structure of a journal selection model [36]

esult of setting certain variables to one. At the same time, other variables that should be zero to ensure hat the multiple choice constraints are not violated because of those which have been set to one, are et to zero, further restricting the number of free variables. In case a particular right-hand side value n a budget constraint) is decreased to less than zero, Kraft and Hill sequentially select variables, cur-ently at one, to set to zero so as to regain feasibility. The final solution is guaranteed to be feasible but, f course, not necessarily optimal.

The second method proposed by Kraft and Hill [54] involves the use of Lagrange multipliers nd will therefore be discussed in a subsequent section on GLM techniques.

The paper by Glover and Klingman [36] also includes a dynamic programming procedure which mploys surrogate constraints [37]. The surrogate constraint strategy consists of replacing the budget onstraints by a single inequality which is a nonnegative linear combination of these constraints. This pproach, in its simplest form, resembles the Lagrange multiplier approach (section 4.4).

A recent paper by Yormark [80] proposes a variant on a forward recursive method first formulated y Greenberg [41]. The approach is similar to that of Gomory and Gilmore [29, 30, 31, 32] in that a for-ward process is used, and that the algorithm takes explicity advantage of the "breakpoints" in the veights. The procedure was formulated for the equality constraint knapsack problem but can be nodified to solve the inequality problem.

We now turn our attention to enumerative integer programming algorithms. As a number of these numerative schemes utilize properties of the associated linear program, i.e., maximize $\sum_i v_i x_i$, subject o $\sum_i w_i x_i \leq W$, and $x_i \geq 0$, with or without upper bounds on the variables, a brief discussion of this roblem is appropriate.

4.2 The Linear Programming Solution to the Knapsack Problem

It is easily shown (see e.g., Dantzig [22]) that if the variables are ordered such that $v_i/w_i \geq v_{i+1}/w_{i+1} \geq \ldots$ $(i=1, 2, \ldots, N-1)$ we obtain the following results:

THEOREM: The optimal solution to the linear program maximize $\sum_i v_i x_i$, subject to $\sum_i w_i x_i \leq W$ nd $x_i \geqq 0$, $i=1, 2, \ldots, N$ is $x_1=\frac{W}{w_1}$, $x_2=x_3=\ldots=x_N=0$ if $v_i>0$ and $x_i=0$, $i=1, 2, \ldots, N$ if

$v_1 \leq 0$. Moreover, if the variables are bounded, i.e., $x_i \leq u_i (i=1, 2, \ldots, N)$, and $w_1 u_1 \leq W$, the optimal solution is found by, starting with $i=1$, setting x_i $(i=1, 2, \ldots, t \leq N)$ equal to u_i until $\sum_{i=1}^{t+1} w_i u_i > W$, and then by setting

$$x_{t+1} = \left(W - \sum_{i=1}^{t} w_i u_i \right) \Big/ w_{t+1}$$

with the remaining variables set to zero. This assumes $v_1, v_2, \ldots, v_{t+1} > 0$. If $v_k \leq 0 (1 \leq k \leq t+1)$, then the optimal linear programming solution is $x_i = u_i$ for $i=1, 2, \ldots, k-1$ and $x_i = 0$ for $i=k, k+1, \ldots, N$.

4.3 Integer Programming Algorithms

4.3.1 Branch and Bound

A branch and bound algorithm proceeds by repeatedly partitioning the class of feasible solutions into mutually exclusive and collectively exhaustive subclasses. The method was first described by Land and Doig [57]. This algorithm was later specialized for the traveling salesman problem by Little, Murty, Sweeney, and Karel [59]. Many variations on the classical version by Land and Doig [57] have appeared since then. An excellent survey of these variations can be found in Salkin [68], and Lawler and Wood [56].

Kolesar [55], and Greenberg and Hegerich [42] describe a branch and bound algorithm in the sense of Little, et al. [59] for the knapsack problem with zero-one variables. In the algorithm by Kolesar, the linear programming subproblems in the "free" (or not fixed) variables are solved using the ratio procedures outlined previously. The ratios are also used to determine the next item to be set to 0 and 1, respectively, which produces successive integer points (nodes); that is, the node whose linear programming solution has the highest functional value is branched from, and the next item to be "fixed" is that free variable with the highest v_i/w_i among the free variables.

Modifications to the algorithm are discussed. The use of additional tests such as lower bound elimination, and additional comparisons between dangling nodes are described. An extension of the algorithm to the multi-dimensional case is also included. It should be noted, that a distinct advantage of this type of algorithm is that computations can be stopped at any time it is felt that results are satisfactory.

4.3.2 Other Enumerative Schemes

Recently, Swart [73] presented a zero-one enumeration scheme for a class of precedence constrained problems of the knapsack type. The general formulation is:

$$\text{maximize} \sum_{i=1}^{m} \sum_{j=1}^{n_i} v_{ij} x_{ij},$$

$$\text{subject to} \sum_{i=1}^{m} \sum_{j=1}^{n_i} w_{ij} x_{ij} \leq W,$$

$$x_{ij} \geq x_{ik} \text{ for some } j, k \text{ and } i = 1, 2, \ldots, m,$$

$$x_{ij} \text{ binary for all } i, j.$$

Swart [53] shows that this problem is equivalent to the formulation

$$\text{maximize} \sum_{i=1}^{m} \sum_{j=1}^{t_i} \bar{v}_{ij} \bar{x}_{ij},$$

$$\text{subject to} \sum_{i=1}^{m} \sum_{j=1}^{t_i} \bar{w}_{ij} \bar{x}_{ij} \leq W,$$

$$\text{and} \sum_{j=1}^{t_i} \bar{x}_{ij} = 1; \; i = 1, 2, \ldots, n,$$

$$\bar{x}_{ij} \text{ binary for all } i, j.$$

Unfortunately, however, there are as many variables $\bar{x}_{ij}$ as there are zero-one solutions to the precedence constraints and each must be found. Problems of this type arise frequently in capital budgeting and are thus of considerable interest.

An enumerative algorithm which successively finds tighter lower and upper bounds is described by Cabot [20]. The procedure applies the Fourier-Motzkin elimination method for solving linear inequalities. Some computational results are reported.

An extension of Balas' additive algorithm [2] specialized for the zero-one knapsack problem which incorporates flexible search enumeration options [45] and heuristics (derived from the linear programming solution) for getting in initial binary solution is described by Guignard and Spielberg [44]. Essentially, the algorithm is an enumeration which incorporates the following options: state enumeration or flexible branching strategies, see [46]), linear programming, the use of Benders' type inequalities [12] (a type of surrogate constraint), inequalities derived from the equivalent group minimization problem (see [40]), and other implicit enumeration criteria as discussed in [58].

4.4 Heuristic Methods and G.L.M. Techniques

Cabot and Hurter [19] present a linear programming approach to the zero-one integer programming solution. Specifically, if we denote the classical zero-one integer program by

(1): maximize cx, subject to $Ax \leq b$, and x binary, we can perturb b by some vector $\epsilon (0 < \epsilon < 1)$ and obtain a linear programming formulation which has the same set of feasible integer solutions as (1), namely

(2): maximize cx, subject to $Ax \leq b(\epsilon) 0 \leq x \leq 1$. Furthermore, if some number N is known, where N is the sum of positive x's in a feasible integer solution to (2), then we can write:

(3): maximize $f(x, y)$, subject to $Ax + Iy = b(\epsilon)$, and $ex = N$, $0 \leq x \leq e$, and the following theorem applies (e is a vector of ones)

THEOREM: A basic feasible solution to (3) is an integer solution if, and only if, every element of the vector y is basic. Applying this to the knapsack problem:

(4): maximize vx, subject to $wx \leq W$, and x binary allows the following formulation. Consider a feasible solution obtained by solving the problem without the integer constraint and setting the frac-

tional part to zero; suppose the value of the objective function for this solution is Z_L. Then we can formulate the following problem.

(5): maximize $f(x_1, y_1, y_2)$, subject to $2wx + y_1 = 2W + 1$, $2vx - y_2 = 2Z_L + 3/2$, $ex = N$, $0 \leq x \leq 1$, and $y_1, y_2 \geq 0$. It can be shown then that any basic feasible solution to (5) with y_1 and y_2 basic is a zero-one solution to (5). The author presents a technique to find the desired feasible solutions and discusses the properties of (5) which allow them to obtain "good" zero-one solutions. The method basically consists of ranking extreme points in (5), then increasing Z_L by an integer amount, reoptimizing for a zero-one solution, etc. Refinements, stopping criteria, and preliminary computational experience are included.

Generalized Lagrangian methods (GLM) were first applied by Everett [26] to problems of optimum allocation of resources. A more detailed specialization for discrete programming problems appears in Brooks and Geoffrion [17], Nemhauser and Ullman [63], and in Shapiro [72]. Since then, many new applications have been formulated. Cord [21] and Kaplan [50] discuss the technique in relation to the Lorie-Savage capital budgeting problem [60]. The author concludes that this procedure, because of its relative simplicity, has certain advantages over other solution techniques for the zero-one problem. Important in this respect is the statement that this conclusion is reached because in certain applications (such as capital budgeting) the constraints are less binding than is indicated by the problem statement. This allows for solutions which "almost meet the constraints." That is, a contraint may be violated by an amount ϵ, where ϵ is small enough to be acceptable, realizing that constraint values usually are crude estimates. Also, the GLM procedure solves several problems successively and hence, solutions to problems with different values for λ and thus different right-hand sides are obtained which may be of interest. These are only obtainable using a post-optimization technique when direct optimization methods are employed. It must be concluded, however, that GLM techniques have serious shortcomings if *exact* solutions to the integer program are required. (The use of GLM in solving nonlinear zero-one programming problems is discussed by Hammer, Rosenberg, and Rudeanu [47]).

Everett's main theorem states that for a given $\lambda \geq 0$, the solution x^* which maximizes the Lagrangian $v(x) - \lambda w(x)$ is also the solution which maximizes $v(x)$ for that constrained problem where $W = w(x^*)$. ($w(x^*)$ may or may not be the right-hand side of interest.) In the case of the knapsack problem the maximization of the Lagrangian involves the maximization of N subproblems, namely $(v_i - \lambda w_i)x_i$ for each i as x_i is binary, the subproblem is solved by setting x_i to zero if $(v_i - \lambda w_i) < 0$, x_i to one if $(v_i - \lambda w_i) > 0$, and $x_i = 0, 1, 2, \ldots$, or u_i if $v_i - w_i = 0$. Observe that the only flexibility in the solution occurs when $v_i - \lambda w_i = 0$ or $\lambda = v_i/w_i$. That is, these and only these "critical" values for λ produce a coefficient equal to zero and thus affect a change in the optimum solution to the Lagrangian for the knapsack problem. Hence, x_i changes, and therefore x^*, $v(x^*)$ and $w(x^*)$ change.

Gulley, Swanson, and Woolsey [43] utilize this information in the formulation of a heuristic algorithm for the zero-one knapsack problem. The algorithm first computes all ratio's v_i/w_i and reorders the variables such that the ratio's are decreasing. It then solves the Lagrangian problem by inspection by listing all solutions. A heuristic is then applied to obtain an improved integer solution x^* until $w(x^*) - b$ is acceptably small.

This algorithm provides "good" solutions which usually suffice in practice. In addition, the availability of an upper bound allows stopping at any point. The authors make the interesting suggestion of using this "Slippery Algorithm" to obtain an excellent starting solution to exact methods.

4.5 Network Approaches

The main results in approaching the knapsack problem from network theory are due to Shapiro [70, 71], who first formulated the knapsack problem as a shortest route problem.

Various algorithms for the shortest route problem are known and will not be discussed in this survey. Discussions on shortest route algorithms can be found in Dreyfus [23], Shapiro and Wagner [69], Shapiro [70], Hu [49], Nemhauser et al. [28], and others. Here, we show that a knapsack problem may be posed as a shortest route problem and include an example of this transformation. Consider the knapsack problem:

$$\text{maximize } \sum_{j=1}^{N} v_j x_j, \text{ subject to } \sum_{j=1}^{N} w_j x_j \leq W,\ x_j \geq 0, \text{ and integer.}$$

Without loss of generality, we assume $w_j \neq w_k$, for $j \neq k$. Let $E_j = \{(i, k)/k - 1 = w_j, 0 \leq i < k \leq W\}$ and consider the directed graph $G = [V, E]$, where $V = 0, 1, \ldots, W$, and $E = \bigcup_{j=1}^{n} E_j$. Let $-v_j$ be the length of each edge in the set of edges E_j. The graph G is acyclic and has the structure shown in Figure 2. Any path in the graph from node O to node W yields a feasible solution to the knapsack

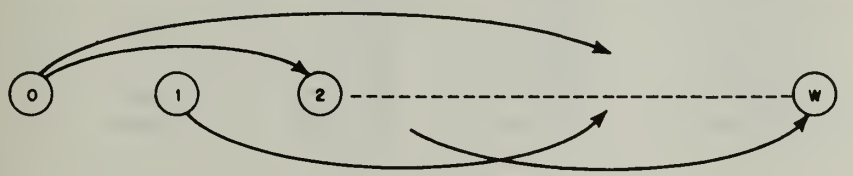
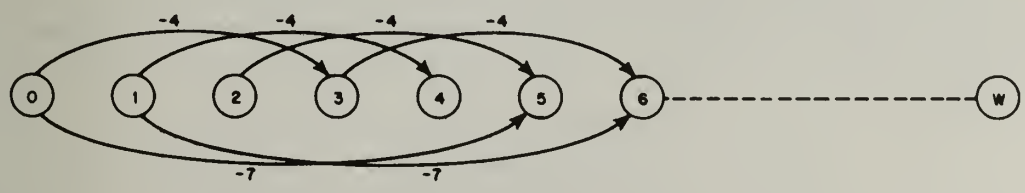


FIGURE 2. The graph corresponding to the knapsack problem

problem by setting x_j equal to the number of edges in E_j in the path. The length of this path is $\sum_{j=1}^{N} v_j x_j$ and as a result the optimal solution to the knapsack problem is given by the shortest path from O to W.

Example (Nemhauser [28])

suppose we have $\begin{cases} w_1 = 3, & v_1 = 4 \\ w_2 = 5, & v_2 = 7 \end{cases}$ then the following arcs would be included in the graph:



The binary knapsack problem cannot be represented this way because of the constraints $x_j \leq 1$. However, the standard dynamic programming recursion for the zero-one knapsack problem corresponds to a graph with NW nodes (as there are N stages and W states in the DP-formulation). The shortest

route approach to the knapsack problem is naturally computationally inefficient because of the usually huge number of nodes and arcs.

5. COMPUTATIONAL RESULTS

5.1 Dynamic Programming Algorithms

Nemhauser and Ullman [65] solved several knapsack problems with bounded variables. The largest one had 50 variables required to be ≤ 2. The costs and weights were randomly generated from a uniform distribution between 0 and 99, and the weight capacity was 2,500. Computation time was 91.2 seconds with the use of an IBM 7094 computer.

Two two-period Lorie-Savage capital budgeting problems were solved by Weingartner and Ness [76]. The two constraint problems were solved by a dynamic programming algorithm with various heuristics. Each problem was tested several times with different heuristic options and budget limitations. The data and computation time are described on the following page. No machine is listed.

N (projects)	28	105	
time (seconds)	1.8 to 731	24 to 317	
w_{1j} (period 1)	0 to 170	0 to 365	weights
w_{2j} (period 2)	0 to 150	0 to 150	
v_j (dollars)	110 to 30,800	70 to 107,200	value
W_1 (period 1)	300 to 600	500 to 3,000	budget
W_2 (period 2)	300 to 600	500 to 3,000	limitation

Gillmore and Gomory [30] have applied their linear programming procedures to solve several cutting stock problems. However, the data and times for the knapsack subproblems that appeared during the computations are included in the overall results. Therefore, we cannot summarize their computational experience.

5.2 Enumerative Algorithms

Kolesar [55] coded a branch and bound algorithm for the knapsack problem with zero-one variables. It was tested on approximately 500 sample problems with 3 to 100 variables. Item weights and values were randomly generated. The algorithm was run until either an optimal solution was obtained or until 3,000 nodes remained active. A set of 160 problems with 25 variables and weight capacity 50 were solved in approximately 0.07 minute each, on an IBM 7094 computer. The weights of the items were such that about half the variables were at value 1 in an optimal solution. Of the 500 problems tested, the algorithm terminated before an optimal solution was reached in 16 problems. The difficulty of the problem naturally varies with the number of variables N and weight limitation W. This is verified in the table below which gives the median number of nodes generated in solving ten problems for each combination of N and W.

N	W				
	25	50	75	100	125
25	176	306	239	168	51
40	185	530	443	562	213
50	1408	509	359	445	495

Cabot and Hurter [19] developed an enumerative algorithm which examines extreme points of a easible region to a linear program derived from the knapsack problem. It was tested on 20 randomly enerated, 20 variable knapsack problems. They report that the maximum number of extreme points examined to confirm optimality was 48. For 10 100-variable problems the corresponding number vas 175. In all cases, the weight capacity was approximately one half the sum of the weights. No running imes or computer make are given.

Cabot [20] describes an enumerative algorithm which successively finds tighter upper and lower ounds on variables. It was tested on problems ranging from 10 to 80 variables using a CDC–3600 com- uter. Each problem size was run 50 times with weights and cost generated from a uniform distribution etween 10 and 110, and weight capacity equal to twice the sum of the weights. The average and naximum solution times (in seconds) are listed below.

N	average time (sec)	maximum time (sec)
10	0.020	0.053
20	0.048	0.191
30	0.098	0.381
40	0.210	1.400
50	0.339	1.890
60	0.415	3.502
70	0.655	3.509
80	0.855	7.221

A branch and bound algorithm for the knapsack problem with zero-one variables was tested by Greenberg and Hegerich [42]. The data was randomly generated. Problems with $W = 100$ and $N = 50$ equired an average solution time of 163 seconds on an IBM 360/67 computer. The authors also report hat several 5,000 variable problems were solved in about 4 minutes.

Guignard and Spielberg [44] applied their search enumeration algorithm to a series of 12 zero-one roblems each with 10 variables. Each was run with three different sets of constraint data and four right- and sides. Variations of the basic enumerative approach were tried on each problem. All problems were solved (no times nor computer type are listed). A number of 75– and 100– variable problems were lso run and did not cause any particular difficulties.

5.3 Lagrangian Multiplier Methods (GLM)

Gulley, Swanson, and Woolsey [43] present a detailed summary of computational results using a Lagrangian algorithm with various heuristics on several sets of problems with different right-hand side values. In contrast to other articles, the authors [43] stress approximate solutions, measured as a percentage of the LP optima under the justification that coefficients usually are estimated. Their algorithm does *not* require a computer, and the main interest lies in sensitivity analysis on the results. The test problems varied in size from 10 to 200 variables. Of the 100 knapsack problems solved, 96 were olved to not less than 98 percent of optimality while in no case a solution was less than 89 percent of he LP-optimum. Execution times appear to be linear with size (number of variables) with the general ule that the Slippery Algorithm performs better as the number of variables and the percentage of variables at upper bound increase both in terms of time and percentage of optimality.

SUMMARY

The survey of computational results may not be exhaustive, but it should give the reader an appreciation of the state of the art. For smaller problems, dynamic programming or enumerative routines usually perform well. Moreover, these algorithms are easily programmed and understood by the user. For larger problems, the application of dynamic programming and enumeration is often constrained by limited computer storage facilities. Here, approximate methods and/or enumeration schemes may prove more useful. It was suggested [43] that approximate methods can be used to obtain a "good" starting solution to the problem after which exact algorithms may be applied. Computations may then be terminated when reasonably good solutions are found. (For example, a good starting solution gives a high lower bound that can be used in an enumerative technique.)

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GENERALIZED PROGRAMMING BY LINEAR APPROXIMATION OF THE DUAL GRADIENT: EQUALITY CONSTRAINTS

J. Franklin Sharp

A.T. & T. Analytical Support Center
New York, N.Y.

Michael H. Wagner

Federal Energy Agency

ABSTRACT

A modified generalized programming procedure is presented for solving concave programming problems with equality constraints. The procedure constructs convenient linear approximations of the gradient of the dual and finds points where the approximating functions vanish. In the quadratic programming case, the procedure is finitely convergent. Global convergence is established for the non-quadratic case. Illustrative numerical examples are included.

INTRODUCTION

Wagner in chapter 15 of [8] applies the generalized programming algorithm to a quadratic programming problem. He observes that the optimal solution is at hand after only two iterations, but several more iterations are required to verify this fact. In this paper we show why this occurs, extend the result, and consider the algorithmic usefulness of the extension.

Generalized programming (*GP*) can be interpreted as a procedure for approximating the dual function by outer linearization. In this paper we show that *GP* also provides convenient linear approximations of the gradients of the dual. These approximations seem to offer the potential for accelerating convergence of *GP*, particularly in the later stages of computation. Our purpose in this paper is twofold. We first intend to demonstrate the algorithmic usefulness of the approximation of the dual gradients for the concave programming case with all equality constraints. Secondly, we present the central theoretical background for the development of an algorithm for the general concave programming problem.

The plan of the paper is the following. The operation of the standard (*GP*) algorithm is first summarized. We next show that in the quadratic programming case with affine equality constraints a simple stopping rule yields finite convergence, and a numerical example is presented. A modification of *GP* is then developed which extends our ideas to the nonquadratic case with affine equality constraints. A numerical example providing an illustrative comparison of *GP* and the modified version is presented.

THE GENERALIZED PROGRAMMING PROCEDURE

Generalized programming (GP) was originally developed by Dantzig and Wolfe (chapter 24 in [1])

for solving the concave programming problem with inequality constraints. The problem we wish to solve is

$$\text{(1)} \qquad \begin{aligned} &\text{Max} && f(x) && f: R^n \to R^1 \\ &\text{s.t.} && g_i(x) = 0 \quad i = 1, \ldots, m && g_i: R^n \to R^1, \\ & && x \in X \end{aligned}$$

where f is strictly concave and X is a convex subset of R^n. We assume for the moment that each g_i is affine (linear plus a constant). It is assumed that an optimal solution x^* exists.

We will need several definitions. First let

$$\text{(2)} \qquad L(x, u) \equiv f(x) - \sum_{i=1}^{m} u_i g_i(x).$$

This is the usual Lagrangian function in which u is an m-vector of dual variables. Secondly, let

$$\text{(3)} \qquad D(u) \equiv \operatorname{Max}_{x \in X} L(x, u)$$

The function D is the dual function and the problem of minimizing $D(u)$ is the dual problem (see for example, Geoffrion [3], Lasdon [4], or Luenberger [5]). Let $x(u)$ be the value of x which solves (3) for a given u, namely,

$$\text{(4)} \qquad x(u) = \arg\max_{x \in X} L(x, u)$$

We assume throughout that f and X have whatever properties that are required to ensure that $x(u)$ is well defined for all u.

Generalized programming solves (1) as follows:

STEP 0: Start with a feasible solution x^o. Let $K = 1$, $\pi^o = 0$ (π is an m-vector), and $w^o = f(x^o)$.

STEP 1: Let $u^K = \pi^{K-1}$. Solve

$$\operatorname{Max}_{x \in X} L(x, u^K)$$

denoting the solution x^K.

If $L(x^K, u^K) = w^{K-1}$, stop (x^K is optimal).

Otherwise, go to Step 2.

STEP 2: Solve

$$\text{Max} \quad \sum_{k=0}^{K} \lambda_k f(x^k)$$

$$\text{s.t.} \quad \sum_{k=0}^{K} \lambda_k g(x^k) = 0$$

$$\sum_{k=0}^{K} \lambda_k = 1$$

$$\lambda_k \geq 0 \qquad k = 0, \ldots, K.$$

Denote the $m+1$ vector of optimal dual variables $(\pi_1^K, \ldots, \pi_m^K, w^K)$. Let $K = K+1$ and go to Step 1.

Several remarks are in order. Step 2 is a linear program which changes only by the addition of a column on successive iterations. The dual variable w^k has as its value the value of the optimal solution of this linear program, and the sequence of w^k's is monotonically nondecreasing. It is always true that $L(x^k, u^k) \geq w^{k-1}$. Finally, by strict concavity of f, a column $(f(x^k), g(x^k), 1)$ may be discarded as soon as it is driven out of the basis in Step 2 (Eaves and Zangwill [2]).

GP is most effective when Step 1 can be executed easily, that is, when the dual function is easily obtained. Special structure can often be exploited. Separability, for example, allows the maximization in Step 1 to be decomposed into small subproblems.

GP is globally convergent for (1). Convergence is not generally finite, however. We show in the next section that convergence is finite in the quadratic case in the following sense. As soon as $\lambda o = 0$ in step 2, the optimal solution is at hand.

THE QUADRATIC PROGRAMMING CASE

Consider the specialization of (1):

(5)
$$\text{Max} \quad f(x) = 1/2x'Qx + c'x$$

$$\text{s.t.} \quad g(x) = Ax - b = 0$$

$$x \in R^n,$$

where Q is a negative definite symmetric $n \times n$ matrix, A is an $m \times n$ matrix, b is an m-vector, and c is an n-vector ($'$ denotes matrix transposition). Note that by virtue of the assumptions that Q is negative definite and that there are no nonnegativity restrictions, a problem with inequality constraints cannot be converted into the proper form by the use of slack variables. Also note that by the weak reverse constraint qualification [6] and the assumption on x^*, optimal dual variables are guaranteed to exist.

We will suppose that (5) is to be solved by GP. Before proving one main result we will need three Lemmas. The first has been previously established (see Corollary 2 on page 427 in Lasdon [4], Lemma 1 on page 313 in Luenberger [5], or Chapter V in Wagner [9]). The validity of Lemma 1 rests upon the assumption of strict concavity of f. Lemma 2 is a well known duality result (see Geoffrion [3], for example).

LEMMA 1: $D(u)$ is a continuously differentiable convex function with gradient

$$\nabla D(u) = -g(x(u)). \tag{6}$$

LEMMA 2: If $D(u)$ achieves its minimum at u^*, $x(u^*)$ is the optimal solution to (1).

LEMMA 3: Let f and g be as in (5). Then

$$x(u) = Q^{-1}A'u - Q^{-1}c \tag{7}$$

and

$$g(x(u)) = AQ^{-1}A'u - AQ^{-1}c - b. \tag{8}$$

PROOF:

$$L(x.\ u) \equiv 1/2x'Qx + c'x - u'[Ax - b]$$

$$\nabla_x L(x,\ u)' = x'Q + c' - u'A.$$

Imposing the condition that $\nabla_x L(x,\ u) = 0$ to obtain the maximum of $L(x,\ u)$ over $x \epsilon X$ yields the result. Q.E.D.

THEOREM 1: Suppose that in the course of solution of (5) by *GP* $\lambda_o = 0$ in Step 2 on the Kth iteration. Then

$$\hat{u}^K \equiv \sum_{k=1}^{K} \lambda_k u^k \tag{9}$$

is the optimal vector of dual variables, that is, $\hat{u}^K$ minimizes $D(u)$. Furthermore,

$$\hat{x}^K \equiv \sum_{k=1}^{K} \lambda_k x^k. \tag{10}$$

is the optimal solution for (5).

PROOF: By the operation of the algorithm

$$\sum_{k=0}^{K} \lambda_k g(x^k) = 0. \tag{11}$$

Since $x^k = x(u^k)$ by definition for $k \neq 0$ and since $\lambda_0 = 0$ by assumption, it follows that

$$\sum_{k=1}^{K} \lambda_k g(x(u^k)) = 0. \tag{12}$$

By (8), $g(x(u))$ is an affine function of u which implies that

$$\sum_{k=1}^{K} \lambda_k g(x(u^k)) = g\left(x\left(\sum_{k=1}^{K} \lambda_k u^k\right)\right), \tag{13}$$

provided that $\sum_{k=1}^{K} \lambda_k = 1$.

By combining (9), (12), and (13), we obtain

$$g(x(\hat{u}^K)) = 0. \tag{14}$$

By Lemma 1, it follows easily that the minimum of $D(u)$ is obtained at the point at which the gradient vanishes, namely for u^* such that $g(x(u^*)) = 0$. Hence, by (14), $\hat{u}^K$ is an optimal vector of dual variables.

By (7), $x(u)$ is an affine function of u. Thus

$$\sum_{k=1}^{K} \lambda_k x(u^k) = x\left(\sum_{k=1}^{K} \lambda_k u^k\right) \equiv x(\hat{u}^K),$$

provided that $\sum_{k=1}^{K} \lambda_k = 1$.

But, as before, $x^k = x(u^k)$ for $k \neq 0$ and $\lambda_o = 0$. Thus

$$\hat{x}^K \equiv \sum_{k=0}^{K} \lambda_k x^k = \sum_{k=1}^{K} \lambda_k x(u^k) = x(\hat{u}^K),$$

which in view of Lemma 2 establishes that $\hat{x}^K$ is optimal to (5). Q.E.D.

Thus in the quadratic case the theorem allows GP to be terminated as soon as $\lambda_o = 0$. This will occur in a finite number of iterations.

THEOREM 2: Suppose $f(x^o) < f(x^*)$. Then $\lambda_o = 0$ in a finite number of iterations.

PROOF: We formulate the linear programming dual of Step 2 of GP:

$$\text{Min } w^K$$

$$\text{s.t. } w^K + \pi^K g(x(u^k)) \geq f(x(u^k)) \qquad k = 1, \ldots, K.$$

$$w^K \geq f(x^o)$$

As previously discussed. the sequence of w^K's is convergent to $f(x^*)$. Therefore, for any $\delta > 0$ there exists some $\tilde{K}$ such that $|f(x^*) - w^K| < \delta$ for $K > \tilde{K}$. Choose $\delta = f(x^*) - f(x^o)$.

Then for every $K > \tilde{K}$, $w^K > f(x^o)$. By the complementary slackness conditions of linear programming,

$$\lambda_o[w^K - f(x^o)] = 0$$

which ensures that $\lambda_o = 0$. Q.E.D.

An interesting sidelight is that termination may be accelerated by the choice of a particularly bad starting solution.

TABLE 1. *Solution of Quadratic Programming Example*

k	u	x_1	x_2	$f(x)$	$g(x)$	$L(x, u)$	w	λ_o	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	$\hat{x_1}$	$\hat{x_2}$	$f(\hat{x})$	$\hat{u}$
0	–	7	0	21	0	–	21	–	–	–	–	–	–	–	–	–	–	–	–	–
1	0	5	4	41	2	41	21	1	0	–	–	–	–	–	–	–	7	0	21	–
2	10	0	−1	−7	−8	73	$31\frac{2}{5}$	0	$\frac{4}{5}$	$\frac{1}{5}$	–	–	–	–	–	–	4	3	39	2
3	$4\frac{4}{5}$	$2\frac{3}{5}$	$1\frac{3}{5}$	$29\frac{12}{25}$	$-2\frac{4}{5}$	$42\frac{22}{25}$	$36\frac{1}{5}$	0	$\frac{7}{12}$	0	$\frac{5}{12}$	–	–	–	–	–	4	3	39	2
4	$2\frac{2}{5}$	$3\frac{4}{5}$	$2\frac{4}{5}$	$38\frac{3}{25}$	$-\frac{2}{5}$	$39\frac{2}{25}$	$38\frac{3}{5}$	0	$\frac{1}{6}$	0	0	$\frac{5}{6}$	–	–	–	–	4	3	39	2
5	$1\frac{1}{5}$	$4\frac{2}{5}$	$3\frac{2}{5}$	$40\frac{7}{25}$	$\frac{4}{5}$	$39\frac{8}{25}$	$38\frac{21}{25}$	0	0	0	0	$\frac{2}{3}$	$\frac{1}{3}$	–	–	–	4	3	39	2
6	$1\frac{4}{5}$	$4\frac{1}{10}$	$3\frac{1}{10}$	$39\frac{19}{50}$	$\frac{1}{5}$	$39\frac{1}{50}$	$38\frac{24}{25}$	0	0	0	0	$\frac{1}{3}$	0	$\frac{2}{3}$	–	–	4	3	39	2
7	$2\frac{1}{10}$	$3\frac{19}{20}$	$2\frac{19}{20}$	$38\frac{159}{200}$	$-\frac{1}{10}$	$39\frac{1}{200}$	$38\frac{99}{100}$	0	0	0	0	0	0	$\frac{1}{3}$	$\frac{2}{3}$	–	4	3	39	2
8	$1\frac{19}{20}$	$4\frac{1}{40}$	$3\frac{1}{40}$	$39\frac{79}{800}$	$\frac{1}{20}$	$39\frac{1}{800}$	$38\frac{399}{400}$	0	0	0	0	0	0	0	$\frac{1}{3}$	$\frac{2}{3}$	4	3	39	2

maximize $f(x) = 10x_1 - x_1^2 + 8x_2 - x_2^2$ $\quad x_1^* = 4$ $\quad f(x^*) = 39$
subject to $g(x) = x_1 + x_2 - 7 = 0$ $\quad x_2^* = 3$ $\quad u^* = 2$

Results of application of GP to a numerical example are presented in Table 1. The optimal solution is obtained on the second iteration. Six further iterations are shown to illustrate the normal operation of GP.

We are not necessarily recommending that GP be used to solve quadratic programming problems with equality constraints. Other methods are available. The result of Theorem 1, however, is central to the material of the following section.

NON-QUADRATIC CASE

The result of Theorem 1 suggests the following simple modification of GP for dealing with the nonquadratic case. Instead of choosing $u^K = \pi^{K-1}$ at Step 1 of each iteration, we propose letting $u^K = \hat{u}^{K-1}$.

The rationale for this modification is provided in the proof of Theorem 1. Suppose that Step 2 of GP has provided a set of $m + 1$ basic variables λ_k (with λ_o nonbasic). Without loss of generality, we assume that these are indexed $1, \ldots, m + 1$. Now consider the following linear approximation to the gradient of the dual function:

$$G(u) = -\sum_{i=1}^{m+1} \lambda_i g(x(u^i)), \tag{15}$$

where $u = \sum_{i=1}^{m+1} \lambda_i u^i$ and $\sum_{i=1}^{m+1} \lambda_i = 1$. Then for the actual values of λ_i provided by Step 2,

$$\sum_{i=1}^{m+1} \lambda_i g(x(u^i)) = 0, \sum_{i=1}^{m+1} \lambda_i = 1.$$

Hence if $\hat{u} = \Sigma \lambda_i u^i$,

$$G(\hat{u}) = 0.$$

The approximation $G(u)$ has the property that it is exact for each of the points u^i, $i = 1, \ldots, m$ provided the points u^i are *affinely independent*. Before presenting Theorem 3 which justifies this remark, we state the following characterization. A set of points $u^i, i = 1, \ldots, m+1$, in R^m are affinely independent if the vectors $(u^i - u^{m+1})$, $i = 1, \ldots, m$, are linearly independent. Affine independence guarantees that any point $u \epsilon R^m$ has a unique representation

$$u = \sum_{i=1}^{m+1} \lambda_i u^i, \sum_{i=1}^{m+1} \lambda_i = 1.$$

(Affinely independent sets are discussed in Rockafellar [7]).

THEOREM 3: Let u^i, $i = 1, \ldots, m+1$ be affinely independent. Then

$$G(u^i) = \nabla D(u^i), \; i = 1, \ldots, m+1.$$

PROOF: u^i has a unique representation

$$u^i = \sum_{j=1}^{m+1} \lambda_j u^i$$

which must be

$$\lambda_i = 1, \; \lambda_j = 0 \text{ for } j \neq i, \; j = 1, \ldots, m+1.$$

Hence

$$G(u^i) = -g(x(u^i)) = \nabla D(u^i). \qquad \text{Q.E.D.}$$

Our modification is implemented simply by computing u^K on each iteration. In the unlikely event that u^i, $i = 1, \ldots, m+1$ are not affinely independent, we experience no computational difficulty. The interpretation of $G(u)$ provided by Theorem 3 is just not available in this case.

The details of the modified procedure follow. It is assumed that $\lambda_o = 0$ on the $K-1$th iteration of the normal GP procedure. The steps then become:

Step M1: $u^K = \hat{u}^{K-1} = \sum_{k=1}^{K-1} \lambda_k u^k$. Solve

$$\text{Max } L(x, u^K)$$

$$x \epsilon X$$

denoting the solution x^K.

If $L(x^K, u^K) = w^{K-1}$, stop (x^K is optimal).
Otherwise, go to Step M2.

Step M2 Solve

$$\text{Max } \sum_{k=1}^{K} \lambda_k f(x^k)$$

$$\text{s.t. } \sum_{k=1}^{K} \lambda_k g(x^k) = 0$$

$$\sum_{k=1}^{K} \lambda_k = 1$$

$$\lambda_k \geq 0 \qquad k = 0, \ldots, K$$

Denote the $m + 1$ vector of optimal dual variables
$(\pi_1^K, \ldots, \pi_m{}^K, w^K)$. $K = K + 1$. Go to Step M1.

In order to guarantee global convergence the modified procedure is embedded in the standard GP by the following amendment. If Step M1 generates a column which would fail to enter the basis at Step M2 (i.e., $L(x^K, \pi^{K-1}) < w^{K-1}$), then a column is generated by the standard GP process. We anticipate that this will rarely be required. Our limited computational experience has substantiated this exception. More computational evidence is needed on this point. On the other hand, if columns are periodically generated by standard GP, the algorithm is "mixed" (in Zangwill's terminology [11] and therefore globally convergent.)

TABLE 2. *Solution of Nonquadratic Example by Generalized Programming*

k	u	x_1	x_2	$L(x, u)$	w	λ		$\hat{x}_1$	$\hat{x}_2$	$f(\hat{x})$
0	—	1.2	1.2	—	15.26667	—	—	—	—	—
1	0	2.8284	1.7321	12.58506	15.26667	$\lambda_0 = 1$	$\lambda_1 = 0$	1.2	1.2	15.26666667
2	0.7138	2.1605	1.1116	13.91740	15.26667	$\lambda_0 = 1$	$\lambda_1 = \lambda_2 = 0$	1.2	1.2	15.26666667
3	2.9363	1.4256	0.6601	11.76824	14.42988	$\lambda_2 = 0.7609$	$\lambda_3 = 0.2391$	1.984782	1.003804	14.00032022
4	1.5582	1.7684	0.8537	13.75512	14.14182	$\lambda_2 = 0.5737$	$\lambda_4 = 0.4263$	1.993338	1.001665	14.00006112
5	1.0828	1.9591	0.9730	13.99365	14.02185	$\lambda_2 = 0.1967$	$\lambda_5 = 0.8033$	1.998768	1.000300	14.00000210
6	0.8876	2.0587	1.0397	13.98800	14.00875	$\lambda_5 = 0.5940$	$\lambda_6 = 0.4060$	1.999562	1.000110	14.00000028
7	0.9829	2.0086	1.0057	13.99973	14.00131	$\lambda_5 = 0.1752$	$\lambda_7 = 0.8248$	1.999937	1.000016	14.00000002
8	1.0328	1.9838	0.9892	13.99902	14.00051	$\lambda_7 = 0.6523$	$\lambda_8 = 0.3477$	1.999975	1.000006	14.00000001

minimize $f(x) = \dfrac{8}{x_1} + x_1 + \dfrac{6}{x_2} + 2x_2$ $\qquad x_1^* = 2$ $\qquad f(x^*) = 14$

subject to $g(x) = x_1 + 4x_2 - 6 = 0$ $\qquad x_2^* = 1$ $\qquad u^* = 1$

$x \epsilon X = \{x | x > 0\}$

TABLE 3. *Solution of Nonquadratic Example by Modified Generalized Programming*

k	u	x_1	x_2	$L(x, u)$	w	λ		$\hat{x}_1$	$\hat{x}_2$	$f(\hat{x})$	$\hat{u}$
0	—	1.2	1.2	—	15.26667	—	—	—	—	—	—
1	0	2.8284	1.7321	12.58506	15.26667	$\lambda_0=1$	$\lambda_1=0$	1.2	1.2	15.26666667	—
2	0.7138	2.1605	1.1116	13.91740	15.26667	$\lambda_0=1$	$\lambda_1=\lambda_2=0$	1.2	1.2	15.26666667	—
3	2.9363	1.4256	0.6607	11.76824	14.42988	$\lambda_2=0.7609$	$\lambda_3=0.2391$	1.984783	1.003804	14.00032022	1.2453
4	1.2453	1.8876	0.9271	13.94867	14.06509	$\lambda_2=0.3997$	$\lambda_4=0.6003$	1.996671	1.000832	14.00001521	1.0329
5	1.0329	1.9837	0.9892	13.99901	14.00903	$\lambda_2=0.0891$	$\lambda_5=0.9109$	1.999507	1.000123	14.00000036	1.0044
6	1.0044	1.9978	0.9985	13.99998	14.00123	$\lambda_2=0.0133$	$\lambda_6=0.9867$	1.999932	1.000016	14.00000000	1.0006

minimize $f(x)=\dfrac{8}{x_1}+x_1+\dfrac{6}{x_2}+2x_2$ $\qquad x_1^*=2$ $\qquad f(x^*)=14$

subject to $g(x)=x_1+4x_2-6=0$ $\qquad x_2^*=1$ $\qquad u^*=1$

$x\epsilon X=\{x|x>0\}$

The standard GP procedure and the modified version are compared on a small numerical example in Tables 2 and 3. The problem being solved requires minimization rather than maximization and the algorithms are appropriately adjusted. The problem is one which might have arisen in the context of determining economic order quantities in the face of a budget constraint. It is seen that in problem solution λ_o equals zero on the third iteration. The modified version demonstrates marked superiority on all subsequent iterations. All calculations were carried out to eight decimal places on an electronic calculator.

SUMMARY

In this paper we have presented a modified version of the Dantzeg-Wolfe generalized programming algorithm for concave programs with all equality constraints. We have shown that the modified procedure utilized linear approximations of the gradient and finds points for which the approximating gradient vanishes. We have also shown that the modified procedure is globally convergent, and finitely convergent for the quadratic programming case.

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A LINEAR PROGRAMMING SOLUTION TO THE GENERALIZED RECTANGULAR DISTANCE WEBER PROBLEM

James G. Morris

Department of Administrative Sciences
Kent State University

ABSTRACT

The Weber Problem generalized to the location of several new points with respect to existing points is formulated as a linear programming problem under the assumption that distances are rectangular. The dual problem is then formulated and subsequently reduced to a problem with substantially fewer variables and constraints than required by an existent alternative linear programming formulation.

Flows may exist between new as well as between new and existing points. Linear constraints can be imposed to restrict the location of new points. Pairwise constraints limiting distances between new points and between new and existing points can also be accommodated.

INTRODUCTION

Consider a set of n distinct points a_j, $j=1, \ldots, n$ in N-dimensional space. Let these points represent the location of existing facilities. The term facilities is used in a generic sense to encompass such entities as machines on a factory floor, warehouses, health clinics, schools, or customer locations. The problem to be considered is the determination of point locations x_i, $i=1, \ldots, m$ for m new or variable facilities with respect to the n existing facilities in order to minimize the sum of transportation costs in the system. It is assumed that costs are proportional to inter-facility distances where the known nonnegative constants of proportionality are respectively p_{ij} and q_{ik} for the distance between new facility i and existing facility j and the distance between new facilities i and k. These constants allow for the location model to reflect the relative importance of inter-facility distances and might be related to multiplicity of trips and/or the amount of flow between two points in a given time period. Letting $d(x_i, a_j)$ denote the distance between the points x_i and a_j the problem can be stated as the minimization of

$$W(x_1, \ldots, x_m) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} \; d(x_i, a_j) + \sum_{i=1}^{m-1} \sum_{k=i+1}^{m} q_{ik} \; d(x_i, x_k). \tag{1}$$

The literature of this problem sometimes referred to as the Generalized Weber Problem, has expanded greatly in the past decade. Literature surveys can be found in [2], [5], and [13].

Frequently when location is to be accomplished on an intraurban basis or on a factory floor travel

between points is constrained to be carried out on a rectangular grid of streets or aisles. The relevant distance function for use in (1) becomes

$$d_r(x_i, a_j) = |x_{1i} - a_{1j}| + |x_{2i} - a_{2j}|,$$

which is here termed the rectangular distance between points $x_i = (x_{1i}, x_{2i})$ and $a_j = (a_{1j}, a_{2j})$ in the plane. This function also approximates certain inter-city road distances if major elements of the road network approach a rectangular grid form. Use of the function assumes that the coordinate axes are oriented so that the direction of movement in the transportation system is always parallel to one or the other of the axes.

The analysis to follow is for two-dimensional problems although extension to N-dimensional problems is straightforward. The resultant problem is to

$$\text{(2)} \qquad \text{minimize } R(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij}(|x_{1i} - a_{1j}| + |x_{2i} - a_{2j}|)$$

$$+ \sum_{i=1}^{m-1} \sum_{k=i+1}^{m} q_{ik}(|x_{1i} - x_{1k}| + |x_{2i} - x_{2k}|),$$

where $x' = \{x_{11}, \ldots, x_{1m}, x_{21}, \ldots, x_{2m}\}$. Bindschedler and Moore [1] developed a graphical solution to the problem in 1961 when $q_{ik} = 0$ for all i and k. In 1963 Francis [4] provided an analytical solution when all $q_{ik} = 0$ and a graphical solution for the case $m = 2$ and $q_{12} > 0$. In 1970 Cabot, Francis, and Stary [2] proposed a network flow solution procedure while Pritsker and Ghare [8] suggested a gradient technique. Both solution methods were for unconstrained problems. In 1971 Wesolowsky and Love [11] showed that $R(x)$ could be minimized using linear programming which enabled the solution of problems with linear locational constraints. Recently Wesolowsky and Love [12] presented an efficient nonlinear approximation approach to minimizing $R(x)$ where any number of linear and/or nonlinear constraints defining a convex feasible region can be included.

This paper describes an alternative linear programming approach for determining an exact minimum of $R(x)$ subject to linear constraints. This alternative formulation substantially reduces the number of variables and constraints and thus the computational effort required to solve the problem as a linear program.

REFORMULATION OF THE PROBLEM

Two properties are now stated which together provide for the restatement of the generalized rectangular distance Weber Problem as a linear programming problem.

PROPERTY 1: An inequality of the form

$$\text{(3)} \qquad \delta \geq \{|x_{1i} - a_{1j}| + |x_{2i} - a_{2j}|\}$$

is equivalent to the following set of inequalities

$$\delta \geq \{+(x_{1i} - a_{1j}) + (x_{2i} - a_{2j})\}$$

$$\delta \geq \{-(x_{1i} - a_{1j}) + (x_{2i} - a_{2j})\}$$

$$\delta \geq \{+(x_{1i} - a_{1j}) - (x_{2i} - a_{2j})\} \tag{4}$$

$$\delta \geq \{-(x_{1i} - a_{1j}) - (x_{2i} - a_{2j})\},$$

where δ, x_{1i} and x_{2i} are variables and a_{1j} and a_{2j} are constants.

Property 1 is easily proven by noting that the dominating force on δ is given by the maximum value of the bracketed terms in (4). Clearly that maximum value will be equal to the bracketed term in (3).

COROLLARY TO PROPERTY 1: Removal of the absolute value signs from an inequality which is the N-dimensional extension of (3) can be accomplished by replacing the original constraint by a set of 2^N inequalities which comprise the N-dimensional extension of (4). For example, in three-dimensional space the replacement of

$$\delta \geq \{|x_{1i} - a_{1j}| + |x_{2i} - a_{2j}| + |x_{3i} - a_{3j}|\}$$

requires a set of eight inequalities to express an equivalent relationship between δ and the weighted rectangular distance between the points (x_{1i}, x_{2i}, x_{3i}) and (a_{1j}, a_{2j}, a_{3j}) without the use of absolute values.

PROPERTY 2: Minimizing $R(x)$ can be restated as

$$\text{minimize} \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} d_{ij} + \sum_{i=1}^{m-1} \sum_{k=i+1}^{m} q_{ik} H_{ik} \tag{5}$$

subject to

(a) $$l_{ij} \geq |x_{1i} - a_{1j}| + |x_{2i} - a_{2j}| \quad i = 1, \ldots, m; \qquad j = 1, \ldots, n$$

(b) $$h_{ik} \geq |x_{1i} - x_{1k}| + |x_{2i} - x_{2k}| \quad i = 1, \ldots, m-1; \qquad k = i+1, \ldots, m.$$

For this and all successive formulations whenever $p_{ij} = 0$ then d_{ij} is to be omitted as a variable of the problem, and whenever $q_{ik} = 0$ then h_{ik} is to be omitted. Since the remaining p_{ij} and q_{ik} are positive it can be seen that an optimal solution to (5) must have the associated variables $d_{ij} = |x_{1i} - a_{1j}| + |x_{2i} - a_{2j}|$ and $h_{ik} = |x_{1i} - x_{1k}| + |x_{2i} - x_{2k}|$.

Properties 1 and 2 allow for the statement of problem (2) as the following linear programming problem:

$$\text{minimize} \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} d_{ij} + \sum_{i=1}^{m-1} \sum_{k=i+1}^{m} q_{ik} h_{ik} \tag{6}$$

subject to

(a)
$$\left.\begin{aligned} d_{ij}+x_{1i}+x_{2i} &\geq +a_{1j}+a_{2j} \\ d_{ij}-x_{1i}+x_{2i} &\geq -a_{1j}+a_{2j} \\ d_{ij}+x_{1i}-x_{2i} &\geq +a_{1j}-a_{2j} \\ d_{ij}-x_{1i}-x_{2i} &\geq -a_{1j}-a_{2j} \end{aligned}\right\} \begin{aligned} &i=1, \ldots, m; \\ &j=1, \ldots, n \end{aligned}$$

(b)
$$\left.\begin{aligned} h_{ik}+x_{1i}-x_{1k}+x_{2i}-x_{2k} &\geq 0 \\ h_{ik}-x_{1i}+x_{1k}+x_{2i}-x_{2k} &\geq 0 \\ h_{ik}+x_{1i}-x_{1k}-x_{2i}+x_{2k} &\geq 0 \\ h_{ik}-x_{1i}+x_{1k}-x_{2i}+x_{2k} &\geq 0 \end{aligned}\right\} \begin{aligned} &i=1, \ldots, m-1; \\ &k=i+1, \ldots, m. \end{aligned}$$

This formulation has $mn+(m^2-m)/2+2m$ variables and $4mn+2\ (m^2-m)$ constraints. The variables are not explicitly restricted to be nonnegative. It has been shown [10] for $m=1$ in $R\ (x)$ that it is only necessary to consider "intersections points" in the convex hull of $\{a_1, \ldots, a_n\}$ as candidates for optimal locations. Recently it has been proven [7] that a certain subset of such points need only be considered for $m \geq 1$. Hence if the origin is shifted to $(\min_j a_{1j}, \min_j a_{2j})$, the optimal coordinates (x_{1i}^*, x_{2i}^*) $i=1, \ldots, m$ will be nonnegative. Since the variables d_{ij} and h_{ik} are distances in the optimal solution and perforce nonnegative, the transformed problem will have an optimal solution with all variables nonnegative. Problem (6) could be solved directly using the algorithm by Gill and Murray [6] which needs no nonnegativity restrictions on the variables. These and other well known techniques for solving (6) are unnecessary since the dual problem which contains fewer constraints is computationally preferable. The dual problem is

(7)
$$\text{maximize} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{s=1}^{4} [(-1)^{p_1(s)} a_{1j} + (-1)^{p_2(s)} a_{2j}] v_{ijs},$$

subject to

(a)
$$\sum_{j=1}^{n} \sum_{s=1}^{4} (-1)^{p_1(s)} v_{ijs} + \sum_{r=1}^{i-1} \sum_{s=1}^{4} (-1)^{p_1(s)+1} u_{ris} + \sum_{k=i+1}^{m} \sum_{s=1}^{4} (-1)^{p_1(s)} u_{iks} = 0 \qquad i=1, \ldots, m;$$

(b)
$$\sum_{j=1}^{n} \sum_{s=1}^{4} (-1)^{p_2(s)} v_{ijs} + \sum_{r=1}^{i-1} \sum_{s=1}^{4} (-1)^{p_2(s)+1} u_{ris} + \sum_{k=i+1}^{m} \sum_{s=1}^{4} (-1)^{p_2(s)} u_{iks} = 0 \qquad i=1, \ldots, m;$$

(c) $\sum_{s=1}^{4} v_{ijs} = p_{ij} \qquad i=1, \ldots, m; \qquad j=1, \ldots, n$

(d) $\sum_{s=1}^{4} u_{iks} = q_{ik} \qquad i=1, \ldots, m-1; \qquad k=i+1, \ldots, m.$

(e) All variables are nonnegative.

The function $p_1(s)$ is $p_1(s)=s-1$ and $p_2(s)$ is the algebraically greatest integer less than or equal to $(s-1)/2$. Certain statements can be made concerning the form of (7). Terms of the form $(-1)^p$ come about due to permutations of signs in the primal problem. The variables v_{ijs} are due to primal distance constraints between fixed facility j and variable facility i. The subscript s refers to the primal constraint (a) in (6) to which the particular dual variable is related. The u_{iks} variables are due to primal distance constraints between variable facilities i and k. In this case the subscript s refers to the primal constraint (b) in (6) to which the particular dual variable is related.

Equality constraints (c) and (d) in (7) can be used to reduce the number of variables. Letting $v_{ij1} = p_{ij} - v_{ij2} - v_{ij3} - v_{ij4} \geq 0$ and $u_{ik1} = q_{ik} - u_{ik2} - u_{ik3} - u_{ik4} \geq 0$ the dual problem is reduced to

(8) maximize $$\sum_{i=1}^{m} \sum_{j=1}^{n} -2[a_{1j}v_{ij2} + a_{2j}v_{ij3} + (a_{1j}+a_{2j})\, v_{ij4}] + (a_{1j}+a_{2j})p_{ij},$$

subject to

(a) $$\sum_{j=1}^{n} 2(v_{ij2}+v_{ij4}) - \sum_{r=1}^{i-1} 2(u_{ri2}+u_{ri4}) + \sum_{k=i+1}^{m} 2(u_{ik2}+u_{ik4})$$

$$= \sum_{j=1}^{n} p_{ij} - \sum_{r=1}^{i-1} q_{ri} + \sum_{k=i+1}^{m} q_{ik} \qquad i=1, \ldots, m;$$

(b) $$\sum_{j=1}^{n} 2(v_{ij3}+v_{ij4}) - \sum_{r=1}^{i-1} 2(u_{ri3}+u_{ri4}) + \sum_{k=i+1}^{m} 2(u_{ik3}+u_{ik4})$$

$$= \sum_{j=1}^{n} p_{ij} - \sum_{r=1}^{i-1} q_{ri} + \sum_{k=i+1}^{m} q_{ik} \qquad i=1, \ldots, m;$$

(c) $v_{ij2} + v_{ij3} + v_{ij4} \leq p_{ij} \qquad i=1, \ldots, m; \qquad j=1, \ldots, n;$

(d) $u_{ik2} + u_{ik3} + u_{ik4} \leq q_{ik} \qquad i=1, \ldots, m-1; \qquad k=i+1, \ldots, m.$

(e) All variables are nonnegative.

This formulation of the problem requires $3mn + 3(m^2 - m)/2$ variables and $mn + (m^2 - m)/2 + 2m$ nontrivial constraints. This represents $mn + (m^2 + m)/2 - 3m$ less constraints and $mn + (m^2 + 3m)/2$

less variables than required by Wesolowsky and Love's formulation. For a problem of locating 5 new facilities with respect to 10 existing facilities formulation (8) has 50 fewer constraints and 70 fewer variables. The dual to their problem is amenable to solution by a bounded-variable algorithm [9]. However, the relative computational efficiency of such a technique to solve the problem is not clear. Hence, discussion of the relative merits of the different linear programs is confined to comparisons of the numbers of variables and constraints.

Constraints (a) in (8) are associated with the variables x_{1i}, $i = 1, \ldots, m$ in (6). Consequently, when problem (8) is solved the dual variables corresponding to those constraints give the optimal first coordinates x_{1i}^*, $i = 1, \ldots, m$ of the variable facilities. Similarly the coordinates x_{2i}^*, $i = 1, \ldots, m$ are given as the dual variables for constraints (b) in (8).

ADMISSIBLE CONSTRAINTS

It may be true that for geographical and/or political reasons, certain regions of the solution space have to be eliminated as possible locations for new facilities. To accomplish this restriction, linear locational constraints can be included. The general form of such constraints is characterized by $Cx \leq b$, where

M = the number of locational constraints
$C = [c_{li}]$ is an $M \times 2m$ matrix
b is an $M \times 1$ vector.

Inclusion of these constraints in the primal problem given by (6) implies that additional variables z_l, say, must be included in the dual problem (7) and in the reduced dual problem given by (8). Specifically, in both problems the term $\sum_{l=1}^{M} b_l z_l$ must be added to the objective function; the term $\sum_{l=1}^{M} c_{li} z_l$ must be added to the left-hand side of constraints (a), and the term $\sum_{l=1}^{M} c_{li+m} z_l$ must be added to the left-hand side of constraints (b).

Upper bounds on rectangular distances between pairs of facilities can also be imposed on the solution. In problem (6) these restrictions correspond to either $d_{ij} \leq \alpha_{ij}$ for the distance between new facility i and existing facility j or $h_{ik} \leq \beta_{ik}$ for the distance between new facilities i and k where $i < k$. The given constants α_{ij} and β_{ik} serve as distance bounds. Such constraints guarantee that the solution will satisfy "covering" requirements. That is, satisfying the minimum sum of weighted distances criterion of location may lead to a situation wherein certain facilities which have a relatively small but positive flow between them are at a great distance apart. It may be that travel distances beyond certain levels between facilities substantially reduce the value of service rendered from one to the other. This may be the case when some trips between points are made under conditions of urgency. Fitzsimmons [3] has noted that when facilities to be located are ambulance bases and distances are weighted by probabilities of incidents the minimization of transportation costs is equivalent to minimizing the mean response time for the ambulance system. However, he describes a characteristic of the ambulance system not thereby considered. When a call is received the requested ambulance may have been dispatched, thus requiring service from a more distant facility. Utilization of the above

nter-facility distance constraints would guarantee that in addition to the primary base certain neighboring bases serving a community secondarily would be within designated maximum distance limitations.

For each constraint $d_{ij} \leq \alpha_{ij}$ included in (6) let e_{ij} be the associated variable in (7). Then (7) must be altered to include the term $-\alpha_{ij}e_{ij}$ in the objective function and the term $-e_{ij}$ must be included on the left-hand side of constraint (c). In defining problem (8) the transformation $v_{ij1} = p_{ij} - v_{ij2} - v_{ij3} - v_{ij4} + e_{ij} \geq 0$ must then be used, Similarly for each constraint $h_{ik} \leq \beta_{ik}$ included in (6) let f_{ik} be the associated variable in (7). Then in (7) the term $-\beta_{ik}f_{ik}$ must be included in the objective function and the term $-f_{ik}$ must be included on the left-hand side of (d). Finally the transformation $u_{ik1} = -q_{ik} - u_{ik2} - u_{ik3} - u_{ik4} + f_{ik} \geq 0$ must be used to define problem (8).

The use of linear locational constraints and maximum distance constraints in (6) changes problem (8) to

$$(8') \quad \text{maximize} \sum_{i=1}^{m}\sum_{j=1}^{n} -2[a_{1j}v_{ij2} + a_{2j}v_{ij3} + (a_{1j}+a_{2j})v_{ij4}] + (a_{1j}+a_{2j}-\alpha_{ij})e_{ij} + (a_{1j}+a_{2j})p_{ij} - \sum_{i=1}^{m-1}\sum_{k=i+1}^{m}\beta_{ik}f_{ik} + \sum_{l=1}^{M} b_l z_l,$$

subject to

$$\text{(a)} \quad \sum_{j=1}^{n} 2(v_{ij2}+v_{ij4}) - e_{ij} - \sum_{r=1}^{i-1} 2(u_{ri2}+u_{ri4}) - f_{ri} + \sum_{k=i+1}^{m} 2(u_{ik2}+u_{ik4}) - f_{ik} - \sum_{l=1}^{M} c_{li}z_l = \sum_{j=1}^{n} p_{ij} - \sum_{r=1}^{i-1} q_{ri} + \sum_{k=i+1}^{m} q_{ik} \qquad i = 1, \ldots, m;$$

$$\text{(b)} \quad \sum_{j=1}^{n} 2(v_{ij3}+v_{ij4}) - e_{ij} - \sum_{r=1}^{i-1} 2(u_{ri3}+u_{ri4}) - f_{ri} + \sum_{k=i+1}^{m} 2(u_{ik3}+u_{ik4}) - f_{ik} - \sum_{l=1}^{M} c_{li+m}z_l = \sum_{j=1}^{n} p_{ij} - \sum_{r=1}^{i-1} q_{ri} + \sum_{k=i+1}^{m} q_{ik} \qquad i = 1, \ldots, m;$$

$$\text{(c)} \quad v_{ij2} + v_{ij3} + v_{ij4} - e_{ij} \leq p_{ij} \qquad i = 1, \ldots, m; \qquad j = 1, \ldots, n$$

$$\text{(d)} \quad u_{ik2} + u_{ik3} + u_{ik4} - f_{ik} \leq q_{ik} \qquad i = 1, \ldots, m-1; \qquad k = i+1, \ldots, m.$$

(e) All variables are nonnegative.

Only those variables e_{ij}, f_{ik}, or z_l are necessary for which there is an associated primal constraint. It is notable that whereas the addition of locational and distance constraints to the original problem implies that constraints are then added to primal linear programming formulations, no constraints are added to (8).

NUMERICAL EXAMPLE

To demonstrate the use of (8') for the minimization of $R(x)$ subject to constraints a sample problem is solved. The locations of three new facilities are sought with respect to five existing facilities. The data drawn from the example in [11] are displayed in Table 1.

TABLE 1

Coordinates of the Existing Facilities

Facility Number	*Coordinates*
1	(2, 3)
2	(4, 2)
3	(5, 4)
4	(3, 5)
5	(6, 7)

Inter-facility Weights

p_{ij}

i \ j	1	2	3	4	5
1	1	1	6	1	6
2	4	1	1	1	1
3	1	1	1	1	1

q_{ik}

i \ k	1	2	3
1	—	1	1
2	—	—	1
3	—	—	—

The constraints imposed on the solution of the problem are:

$$x_{13}+x_{23} \leq 3$$

$$d_{15}=|x_{11}-6|+|x_{21}-7| \leq 2$$

$$h_{23}=|x_{12}-x_{13}|+|x_{22}-x_{23}| \leq 1.$$

Figure 1 illustrates the problem. Note that the third constraint does not lend itself to graphical exposition.

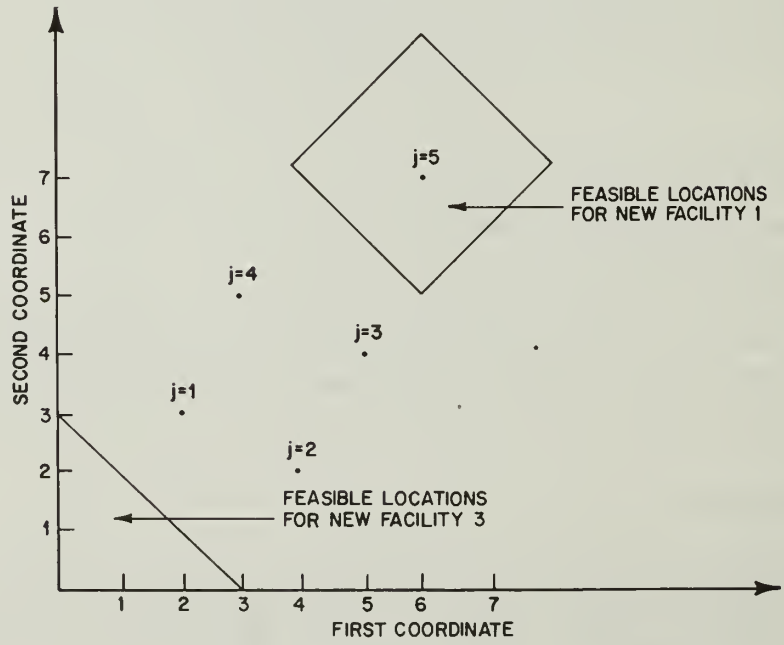

FIGURE 1. The example problem

The problem as given by (8′) with consideration of the three constraints is:

$$\text{maximize } -4v_{112}-6v_{113}-10v_{114}-8v_{122}-4v_{123}-12v_{124}-\ldots$$

$$-12v_{352}-14v_{353}-26v_{354}+11e_{15}-f_{23}-3z_1,$$

subject to

$$2v_{112}+2v_{114}+\ldots+2v_{152}+2v_{154}-e_{15}+2u_{122}+2u_{124}+2u_{132}+2u_{134}=17;$$

$$2v_{212}+2v_{214}+\ldots+2v_{252}+2v_{254}\qquad -2u_{122}-2u_{124}+2u_{232}+2u_{234}-f_{23}=8;$$

$$2v_{312}+2v_{314}+\ldots+2v_{352}+2v_{354}\qquad -2u_{132}-2u_{134}-2u_{232}-2u_{234}+f_{23}+z_1=3;$$

$$2v_{113}+2v_{114}+\ldots+2v_{153}+2v_{154}-e_{15}+2u_{123}+2u_{124}+2u_{133}+2u_{134}=17;$$

$$2v_{213}+2v_{214}+\ldots+2v_{253}+2v_{254}\qquad -2u_{123}-2u_{124}+2u_{233}+2u_{234}-f_{23}=8;$$

$$2v_{313}+2v_{314}+\ldots+2v_{353}+2v_{354}\qquad -2u_{133}-2u_{134}-2u_{233}-2u_{234}+f_{23}+z_1=3;$$

$$v_{112}+v_{113}+v_{114}\leq 1;$$

$$v_{122}+v_{123}+v_{124}\leq 1;$$

$$\ldots$$

$$v_{152}+v_{153}+v_{154}-e_{15}\leq 6;$$

$$v_{212}+v_{213}+v_{214}\leq 4;$$

$$\ldots$$

$$v_{352}+v_{353}+v_{354}\leq 1;$$

$$u_{122}+u_{123}+u_{124}\leq 1;$$

$$u_{132}+u_{133}+u_{134}\leq 1;$$

$$u_{232}+u_{233}+u_{234}-f_{23}\leq 1.$$

All variables are nonnegative.

There are 57 variables in this problem and 24 nontrivial constraints. The solution is to locate new facilities 1, 2, and 3 at the coordinates (5,6), (2,2), and (2,1), respectively, that yield $R(x)=104$ units.

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SCHEDULING WITH EARLIEST START AND DUE DATE CONSTRAINTS ON MULTIPLE MACHINES*

Paul Bratley, Michael Florian, Pierre Robillard †

University of Montreal
Montreal, Canada

ABSTRACT

We consider the scheduling of n jobs on m identical machines when the jobs become available for processing at ready times a_i, $a_i \geq 0$, require d_i time units for processing and must be completed by times b_i for $i = 1,2, \ldots n$. The objective chosen is that of minimizing the total elapsed time to complete all jobs subject to the ready time and due date constraints. preemption is not allowed. We present a multi-stage solution algorithm for this problem that is based on an implicit enumeration procedure and also uses the labelling type algorithm which solves the problem when preemption is allowed.

1. INTRODUCTION

In earlier work [2], [3] we considered the scheduling of n jobs on a single machine where the jobs become available for processing at ready times a_i, $a_i \geq 0$, require d_i time units for processing, and must be completed by times b_i for $i = 1,2, \ldots, n$. In [2] we searched for the feasible schedule that minimizes total elapsed time and in [3] for the feasible schedule that minimizes the maximum lateness. The latter problem was also studied by Baker and Su [1] and Dessouky and Margenthaler [4]. In this paper we consider a similar problem which differs from the above by the presence of m identical machines. The objective chosen is that of minimizing the total elapsed time to complete all jobs subject to the ready time and due date constraints. Preemption is not allowed. We shall present a multi-stage solution algorithm for this problem that is based on an implicit enumeration procedure and also uses the "labelling" type algorithm of [2] which solves the problem when preemption is allowed.

2. THE ENUMERATION OF ALL SEQUENCES OF N JOBS ON M MACHINES

We shall approach the presentation of the solution algorithm by showing first the way to enumerate without repetition all the possible arrangements of n jobs on m identical machines.

The enumeration is accomplished by generating a tree which contains two types of nodes, denoted in our example by circles and squares. An arrangement of the jobs is identified in the tree by a path from the root to a terminal node of the tree. If the path passes through node (i), then this means that "job i is scheduled on the current machine", while if the path passes through node $[i]$, then "job i is

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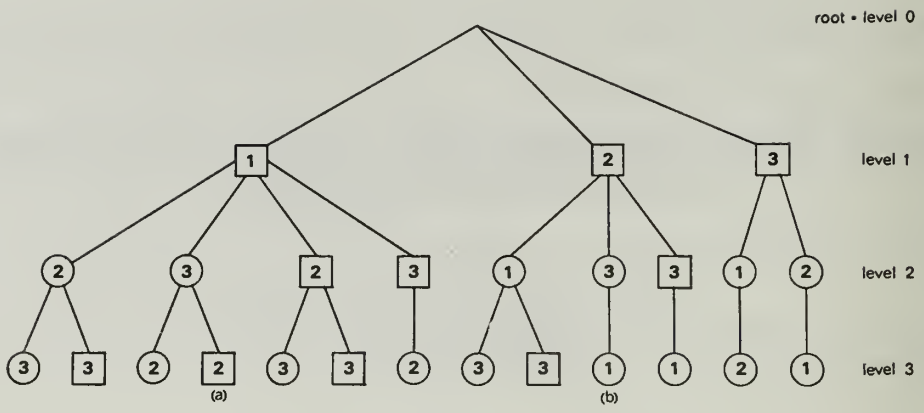

FIGURE 1.

scheduled on a new machine, which becomes the current machine". To illustrate these notions, we give in Figure 1 the tree which enumerates all possible arrangements of three jobs on three machines. For instance, the path from the root to terminal node (a) corresponds to the arrangement in which job 1 is first on the first machine, job 3 follows on the same machine, job 2 is first on the second machine, and (by implication) machine 3 is not used; the path from the root to (b) corresponds to the arrangement in which jobs 2, 3, and 1 are in that order on the first machine, while machines 2 and 3 are unused.

It may easily be verified that the number of arrangements of n jobs on exactly m machines is $\binom{n-1}{m-1}\frac{n!}{m!}$, so that the number of arrangements on m machines or less is

$$a[n, m] = n! \sum_{i=1}^{m} \binom{n-1}{i-1} \Big/ i!$$

In the example

$$a[3, 3] = 6\left\{\binom{2}{0} + \frac{1}{2}\binom{2}{1} + \frac{1}{6}\binom{2}{2}\right\} = 13.$$

The tree that we have described above can be constructed by using the following set of rules, where we have numbered the levels 0 (for the root), 1, 2, . . . n (terminal level):

(1) Level 0 contains only the dummy root node;

(2) Level 1 contains the nodes $\boxed{1}, \boxed{2}, \ldots, \boxed{n}$;

(3) A path from level 0 to level i, $1 \leq i < n$, may be extended to level $i+1$ by any of the nodes $\textcircled{1}, \textcircled{2}, \ldots, \textcircled{n}, \boxed{1}, \boxed{2}, \ldots, \boxed{n}$ provided rules (4) to (6) are observed;

(4) Neither $\textcircled{k}$ nor $\boxed{k}$ may be used to extend a path in which either $\textcircled{k}$ or $\boxed{k}$ already occurs;

(5) $\boxed{k}$ may not be used to extend a path which already contains some node $\boxed{l}$ with $l > k$.

(6) No path may be extended in such a fashion that it contains more than m square nodes.

Next we show that the rules given generate all possible arrangements without repetition. It is clear that all paths which are started can be extended in at least one way so as to reach level n, and also that no path can be extended beyond level n.

Suppose we are given an arrangement of n jobs on m or less identical machines. Since the machines are identical we may suppose without loss of generality that if k_i is the first job scheduled on machine i, and if k_j is the first job scheduled on machine j, then $i < j = k_i < k_j$, since otherwise we can simply renumber the machines. Now it is easy to see that the arrangements are in 1 to 1 correspondence

with the paths from level 0 to level n in the tree, using the interpretation of round and square nodes given above. Therefore generating the tree is equivalent to enumerating without repetition all the arrangements of n jobs on m identical machines.

The procedure outlined is simple to program, and is the basis of the branch and bound portion of our algorithm. It is convenient to denote round nodes by negative numbers and square nodes by positive numbers, so that a path printed out in the form 1–4–5, 3–6, 2–7, for instance shows rather elegantly both the order of the jobs and the grouping on machines.

This basic algorithm is augmented by the "bounding" part of the "branch and bound" so that only a part of the tree is generated explicitly. Due to the nature of the problem we also use exclusion rules which further reduce the tree generated explicitly.

3. BOUNDING AND EXLUSION RULES

First, we reorder the jobs in increasing order of the ready times such that i precedes j if $a_i \leq a_j$.

We compute a lower bound on the completion time by relaxing the condition that no preemption is allowed. The solution for the relaxed problem is obtained by using the max-flow type "labeling" algorithm given in Reference [2]. In addition, we may tighten the due dates as soon as a feasible solution has been constructed. The value of a solution is the maximum of the completion times of the last jobs on the machines used (we shall call the job which determines this value the *critical job*). Let the solution value be t_{fin}. Then we reduce all due dates b_i to be at most $t_{fin} - 1$.

We present next some properties of this sequencing problem which are used to cut down the tree search by excluding from consideration certain parts of the tree. The aim is to recognize conditions which indicate that permutation of jobs in certain parts of the schedule is unnecessary, since the solution cannot be improved.

(i) *Exclusion:* Suppose that a solution is found and the critical job is at level j. Then try the next alternative at level j immediately, without trying to permute jobs $j+1, \ldots, n$. Clearly, the solution cannot be improved by changing the sequence after job j.

(ii) *Exclusion:* Suppose we are working on machine k, which is available at time t_k. The set $\bar{S}$ of jobs is still not scheduled. Then we can abandon the attempt to extend the schedule if

$$\max_{i \in \bar{S}} b_i - t_k + (m-k)\ \{\max_{i \in \bar{S}} b_i - \min_{i \in \bar{S}} a_i\} < \sum_{i \in \bar{S}} d_i.$$

(iii) *Exclusion:* Suppose again that we are working on machine k, which is available at time t_k. Job i should not be considered for extending the sequence if there exists some other job $j \in \bar{S}$ such that

$$\max(a_j, t_k) + d_j \leq b_j \text{ (i.e. } j \text{ is feasible on } k)$$

and

$$\max(a_j, t_k) + d_j \leq a_i \text{ (i.e. } j \text{ can be scheduled on } k \text{ before job } i \text{ without holding it up).}$$

(iv) *Exclusion:* Suppose again that we are working on machine k which is available at time t_k.

Then if $t_k \leq \min_{i \in \bar{S}} a_i$ there is no point in considering other permutations of the jobs already assigned to machine k.

(v) *Solution Split:* Recall that the jobs are sorted in increasing order of a_i. Suppose that the first i jobs have been scheduled and that the machines are available at times $t_1, t_2, \ldots, t_m$. Without loss of generality, suppose that $t_1 \leq t_2 \leq t_3 \leq \ldots \leq t_m$. Then the solution decomposes before job $i+1$ if

$$t_j \leq a_{i+j}, j = 1, 2, \ldots \min(n-i, m),$$

for in this case, any arrangement of jobs $i+1, i+2, \ldots, n$ may be used to extend the existing schedule without interference (subject only to a possible rearrangement on the machines).

We call the set of jobs scheduled between two successive solution splits a *block*. Note that a block necessarily consists of consecutively numbered jobs, because of the ordering chosen for the jobs. If the above rule is applied generally, the critical job is not necessarily in the last block. To avoid this inconvenience, we do not recognise a solution split unless $n - i \geq m$, a condition which ensures that the critical job must be in the last block.

With this extra condition, it is apparent that jobs once scheduled in a block other than the last can be excluded from further consideration, and that only the jobs in the last block of a schedule need to be permuted to attain an optimum.

(vi) In addition, the following heuristic rule is used in the tree generation in an attempt to find a good initial feasible solution: always schedule the first available job on the first available machine; if several jobs are available, break ties by choosing the one with the smallest b_i; if a tie still occurs, select the job with the longest d_i.

4. THE COMPOSITE IMPLICIT ENUMERATION ALGORITHM

We combine the enumeration procedure with the bounding, exclusion and heuristic rules to obtain a composite implicit enumeration algorithm that consists of several phases. The algorithm is shown in block diagram form in Figure 2 and is described in detail below:

① *Initialization*

Sort the jobs in increasing order of ready times a_i.

② *Determine the smallest number of machines required to obtain a feasible solution with preemption*

2a. Set $m = m_0$ (m_0 is the number of machines given initially)

2b. Solve with preemption allowed using the algorithm described in [2].

2c. If a feasible solution with preemption is obtained, save its value T_{split} and continue; otherwise set $m = m + 1$ and return to 2b.

2d. Set Lower Bound $= T_{split}$.

③ *Implicit Enumeration*

3a. Set F, the number of frozen jobs, and i, the index of jobs, to 0.

3b. Set $i = i + 1$.

3c. If there is a solution split before job i, set $F = i - 1$.

3d. Try to add job i to the schedule using the heuristic mentioned above. If it will not fit, go to 3h; otherwise continue.

3e. If $i < n$ go to 3b; otherwise continue.

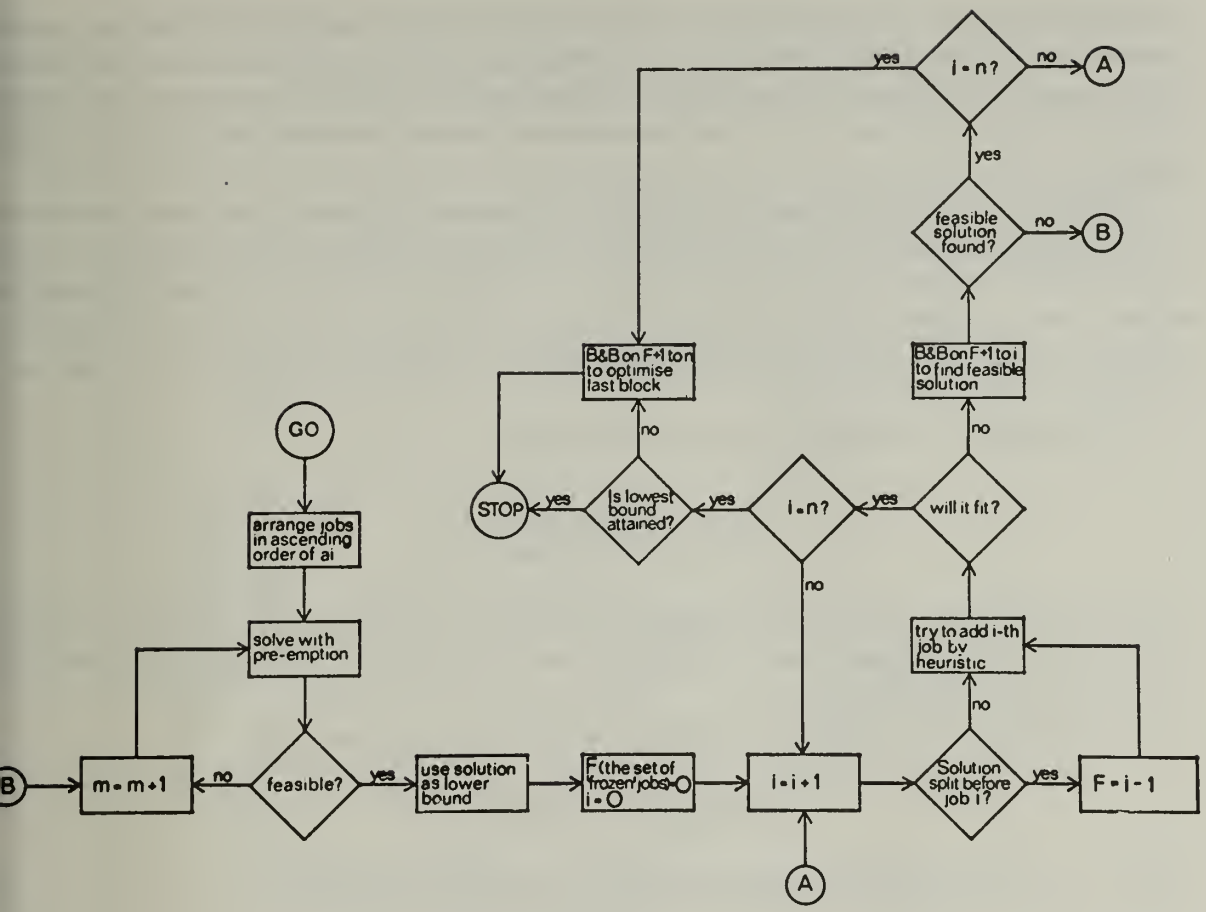

FIGURE 2.

3f. If the value of the new solution is equal to the lower bound, STOP, the solution is optimal; otherwise continue.

3g. Generate a tree using the exclusion rules for jobs $F+1, \ldots, n$ to optimise the last block and then STOP, the solution thus obtained is optimal.

3h. Generate a tree using the exclusion rules for jobs $F+1, \ldots, i$ to find a feasible solution for just these jobs. If a feasible solution is found, go to 3e; otherwise set $m = m+1$ and go to 2b.

Two points are worth making. First, the version of the algorithm presented here always finds a feasible solution, if necessary by increasing the number of machines used. It could of course easily be adapted simply to abandon the computation if an infeasible problem was found. Secondly, it will be noticed that the algorithm proceeds as far as possible using a heuristic, falling back on a full branch and bound solution only when it is blocked, and *continuing on using the heuristic* after a block has been circumvented. This approach gives much better computation results than simply abandoning heuristic methods at the first sign of trouble.

5. COMPUTATIONAL EXPERIENCE

The evaluation of the numerical performance of the implicit enumeration algorithm described in this paper is a difficult task. When the number of jobs and number of machines is large the size of

the enumeration tree is very large and the solution may not be obtained in a reasonable time on a modern computer. In addition, the difficulty of the problem varies with the distribution of the ready times, due dates, and job durations. In the following we describe our computational experience with certain classes of problems and point out certain characteristics that are relevant in the eventual use of the algorithm.

We have tested the algorithm on a series of randomly generated problems. The relevant parameters are n, the number of jobs, m the number of machines, $a_{\max}$, the largest of the ready times, $d_{\max}$, the largest of the job durations and $s_{\max}$, the largest of the slack times, $s_i = b_i - d_i - a_i$. The a_i's were selected from a uniform distribution between 0 and $a_{\max}$, the d_i's were selected from a uniform distribution between 1 and $d_{\max}$ and the slack values s_i from a uniform distribution between 0 and $s_{\max}$. Evidently $b_i = a_i + d_i + s_i$.

For each choice of parameter we computed the value

$$L = \frac{d_{\max} \times n/2}{(a_{\max} + d_{\max} + s_{\max}) \times m}.$$

In this expression, the numerator is a rough measure of the total processing time required to complete all the jobs and the denominator approximates the usable time of the m machines. We shall refer to L as the *load factor* of the problem.

First we generated 9 groups of 10 problems each. Four of these groups had

$$n = 15,\ m = 2,\ a_{\max} = 50,\ s_{\max} = 15,$$

while $d_{\max}$ varied between 10 and 25 in steps of five. The other five groups had

$$n = 25,\ m = 3,\ a_{\max} = 100,\ s_{\max} = 15$$

while $d_{\max}$ varied between 10 and 30 in steps of 5. The results for these problems are given in Table 1.

This table gives the median time in seconds for solving each group of 10 problems, and also the number of problems unsolved in each group: in the interests of economy, the attempt to solve a problem was abandoned if the solution had not been found after 50,000 branches, or about 2.5 seconds of computation.

$d_{\max}$	$n=15$, $m=2$, $a_{\max}=50$, $s_{\max}=15$	$n=25$, $m=3$, $a_{\max}=100$, $s_{\max}=15$
10	0.12	0.16
15	0.22(1)	0.47
20	0.63(3)	1.47
25	1.21(4)	1.60(4)
30		2.92(4)

TABLE 1. *Value of Median Time in Seconds for the Solution of 10 problems. The numbers in brackets indicate the number of unsolved problems out of 10.*

igure 3 is a plot of the median of solution times against the index L. As may be anticipated intuitively, ıese times increase with L, which is an index of difficulty of the problem. It may also be seen that the ifficulty of the problem increases with the number of jobs. Further experimentation with a larger umber of jobs would have been too costly.

A second series of tests was performed with the use of 10 groups of 10 problems each, generated ı a similar manner. The parameters used were $n = 25$, $a_{\max} = 100$, $d_{\max} = 25$, $s_{\max} = 15$, and m varied ·om 1 to 10. The results are given in Table 2. These median solution times are plotted against m in ıe graph of Figure 4.

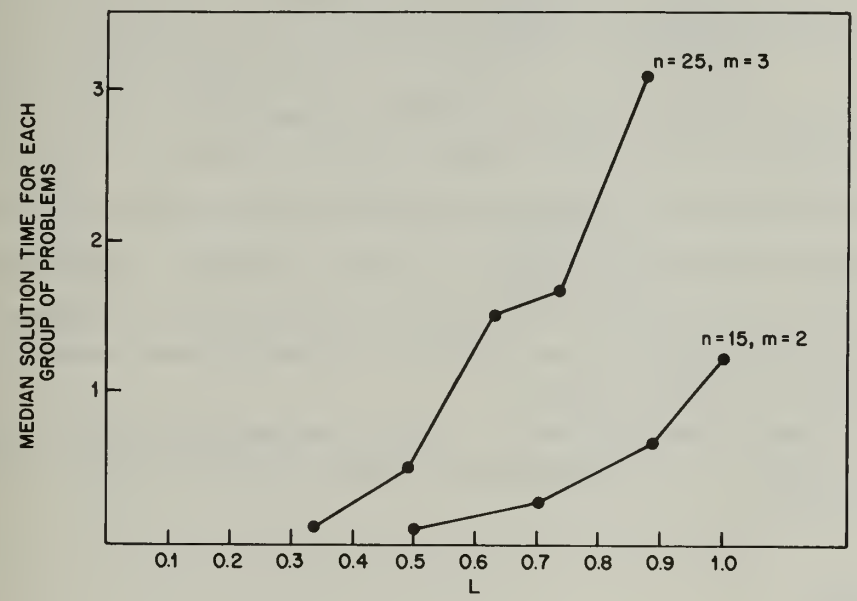

FIGURE 3.

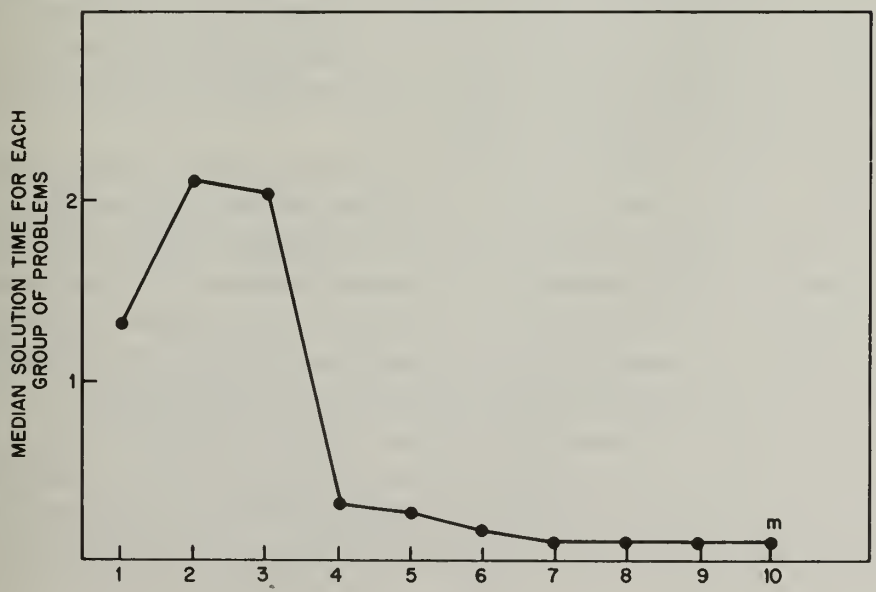

FIGURE 4.

m	Median Solution Time (sec)	Unsolved Problems out of 10
1	1.36	2
2	greater than 2.008	5
3	2.002	3
4	0.32	4
5	0.27	2
6	0.18	–
7	0.13	1
8	0.13	1
9	0.13	–
10	0.13	–

TABLE 2.

The shape of that graph is not surprising since it reflects the fact that the number of arrangements of n jobs on exactly m machines, $\binom{n-1}{m-1}\frac{n!}{m!}$, is a function of m that has a mode between 1 and n at the smallest integer k that satisfies $n < k^2 + 2k$. Here the mode is at $k = 5$. The mode on the graph occurs earlier than this, probably because, although the number of arrangements is increasing still with m, the load factor is rapidly decreasing.

The code we used for these experiments was programmed using the optimising FORTRAN compiler on a CDC CYBER 74. All the times given are in seconds of CPU time, and do not include compilation or input and output.

6. CONCLUSION

We have presented an algorithm for the solution of the problem of minimising the total elapsed time of a schedule on several machines. It will be evident that the algorithm, with trivial modifications, could also serve for minimising the number of machines needed. In the limited number of experiments which we carried out, the algorithm found an optimum solution 136 times out of 170 in about 2 seconds of computation or less. This is a success rate of 80 percent.

It may be asked whether an 80 percent successful algorithm is useful or not. Our reply would be very firmly positive. It seems clear to us that any algorithm whatsoever for scheduling a reasonable number of jobs, say 20, is sooner or later going to hit a pathological example which will demand the evaluation of something like 20! possibilities. No computing machine in the world is capable at the moment of coming within orders of magnitude of this amount of computation. We therefore feel that it is much more intelligent to develop a battery of possible lines of attack with a reasonable probability of success, than it is to demand a unique algorithm capable of resolving any problem.

Viewed from this point, the present algorithm has several useful features. First, it is "fail-safe," which is to say that although it may not find an optimum, it will not return an erroneous value; secondly, the preliminary calculation performed allowing preemption provides a lower bound to the optimum; and thirdly, the best solution (if any) found by the main algorithm provides both an upper bound to the optimum and a feasible solution.

The basic method of enumerating all possible ways of scheduling several jobs on several machines can also clearly be applied to a variety of problems other than the one discussed here.

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SENSITIVITY OF THE (Q, R) INVENTORY MODEL TO PENALTY COST PARAMETER

Chandrasekhar Das

University of Saskatchewan
Saskatoon, Canada

ABSTRACT

This paper treats an approximate continuous review inventory model with backlogging of excess demand and stochastic leadtime. The major result derived is that the behavior of the optimal order size with respect to the shortage cost parameter is determined solely by the "conditional mean residual life" function corresponding to the leadtime demand distribution. Some minor results and illustrative examples are also included.

I. INTRODUCTION

The purpose of this paper is to study the sensitivity of the optimal order size, the optimal reorder point and the minimal expected total inventory cost for a steady-state continuous review inventory model. Specifically, our attention is focused on the monotonicity of the above-mentioned quantities as functions of the penalty cost parameter. Similar issues are discussed in a previous paper by Kaplan [3]. The present paper, however, contains some new results in addition to improving some of the results of [3] by following a different approach. The model underlying our analysis is not exact, but is found to approximate the real situation quite satisfactorily. The model is discussed fully in standard textbooks, e.g., Hadley and Whitin [1] and Wagner [5]. Nevertheless, the main features of the model are stated below for the sake of completeness.

II. THE MODEL

It is assumed that the inventory position is reviewed continuously and a fixed quantity Q is ordered as soon as the net inventory (on hand minus backorders) drops to the reorder point R which is strictly positive. The planning horizon is 1 year and all demands in excess of stock on hand are fully backordered. But a penalty cost is charged at the rate of π per unit backordered. Leadtime (days) and demand per unit time (day) are strictly positive, independent random variables. However, it is assumed that there is not more than one order outstanding at any point in time. The symbols used in the sequel are as follows:

μ = expected annual demand
L = demand during leadtime
K = fixed cost per order
H = holding cost per unit per year

$\phi(L)$ = probability density function of L
$\Phi(L)$ = cumulative density function of L
$\Phi^c(L) = 1-\Phi(L)$
$G(p)$ = upper $100p$ percent point of ϕ so that $\Phi^c(G(p)) = p$
λ = expected value of L.

For simplicity we also assume that backorders can be neglected in computing the average stock on hand and that the probability of L exceeding $R+Q$ is small. The expected total cost (per year) for the above system can then be approximated by:

$$T(Q,R) = (K\mu/Q) + H\{(Q/2) + R - \lambda\} + \pi\mu B/Q, \text{ where} \tag{1}$$

$$B = B(R;\phi) = \int_R^\infty (L-R)\,\Phi(L)\,dL. \tag{2}$$

III. ANALYSIS

Equating the partial derivatives of T to zero it is easily seen that the optimal values of R and Q must satisfy the following pair of simultaneous equations:

$$\Phi^c(R) = HQ/\pi\mu \tag{3}$$

$$Q^2 = (2\mu/H)\cdot(K+\pi B). \tag{4}$$

Denoting the optimal values of R and Q by $R^* = R^*(\pi)$ and $Q^* = Q^*(\pi)$ the following relations are then easily derived from (3):

$$Q^*/\pi = (\mu/H)\Phi^c(R^*) \tag{5}$$

$$\pi = (H/\mu)\{Q^*/\Phi^c(R^*)\} \tag{6}$$

$$R^* = G(HQ^*/\pi\mu). \tag{7}$$

Let us now define

$$\beta = \beta(R;\phi) = B(R;\phi)/\Phi^c(R) \tag{8}$$

and write $B^* = B(R^*;\phi)$, $\beta^* = \beta(R^*;\phi)$. Then by substituting (6) for π we get from (4)

$$Q^{*2} = (2\mu/H)\{K + (HQ^*/\mu)\beta^*\}. \tag{9}$$

Writing $f(x) \sim g(x)$ to mean that $f'(x)$ and $g'(x)$ are of the same sign *for all* x, our main results may then be stated as follows:

LEMMA 1: $R^*(\pi) \sim \pi/Q^*(\pi)$ for all ϕ.

Since G is monotone nonincreasing in its argument for any ϕ, it follows from (7) that as functions of

·, R^* and Q^*/π are inversely related. Hence the result.

THEOREM 1: $Q^*(\pi) \sim \beta(R^*(\pi))$ for all ϕ.

PROOF: It follows from (9) by transposition that Q^* is a root of the equation

10) $$Q^2 - 2\beta^* Q - 2K\mu/H = 0.$$

Since the negative root of (10) is not admissible, we claim that Q^* has the following structural epresentation

11) $$Q^* = \beta^* + \{\beta^{*2} + 2K\mu/H\}^{1/2}.$$

The desired result follows immediately from (11) since $\beta^* > 0$.

LEMMA 2: $\dfrac{B^*}{Q^*} \sim \dfrac{1}{\pi}$ for all ϕ.

This result is proved in [3]. The definition of π given here differs slightly from that of [3], but the line of argument easily carries over to this case.

THEOREM 2: $\dfrac{Q^*(\pi)}{\pi} \sim \dfrac{1}{\pi}$ for all ϕ.

PROOF: If $Q^*(\pi) \sim 1/\pi$, then the assertion follows immediately. Therefore, assume that $Q^*(\pi) \sim \pi$. Since $Q^*(\pi) > 0$, this implies $\pi Q^*(\pi) \sim \pi$ and that $\pi Q^*(\pi) > 0$. Hence, dividing both sides of (4) by $\pi Q^*(\pi)$ we get

$$\frac{Q^*(\pi)}{\pi} = \left(\frac{2\mu}{H}\right) \cdot \left\{\frac{K}{\pi Q^*(\pi)} + \frac{B^*}{Q^*(\pi)}\right\}.$$

Note that both terms within the curly brackets are decreasing in π. The assertion, therefore, holds.

COROLLARY 1: $R^*(\pi) \sim \pi$ for all ϕ.

This result follows directly from LEMMA 1 and THEOREM 2.

COROLLARY 2: If $\beta(x)$ is monotone nonincreasing (or nondecreasing) in x, then so is $Q^*(\pi)$ as a function of π. In particular, if $\beta(x)$ is constant than Q^* is independent of π and R^*.

Since $R^*(\pi) \sim \pi$, monotonicity of β in its argument implies that $\beta(R^*(\pi)) \sim \beta(\pi)$. The stated result then follows directly from Theorem 1 and equation (11).

The importance of the above result is that the monotonic behavior of $Q^*(\pi)$ with respect to π for any leadtime demand distribution ϕ can be determined, often analytically, from the monotonicity of the function $\beta(x; \phi)$ in x. We may stress that a knowledge of R^* is not necessary for this purpose. The task of testing the monotonicity of Q^* is, therefore, made simple by Corollary 2 due to the fact that any parameter not related to the demand or the leadtime distribution may be ignored. If, however, β is not monotonic then a knowledge of the behavior of $Q^*(\pi)$ in a given range of π requires determination

of $R^*(\pi)$. The result of Theorem 2 which is independent of $R^*(\pi)$ shall be helpful in such cases. Lastly, it is interesting to note that $\beta(x;\phi)$ is nothing but the conditional mean residual life of an item (from the population of ϕ) which has survived x units of time.

The previous results pertain to the individual behaviors of Q^* and R^*. It is of some interest to know the joint behavior of these two quantities as functions of π. The following theorem provides some useful information in this regard.

THEOREM 3: The sum Q^*+R^* is an increasing function of π for all ϕ.

PROOF: It is evident from (1) that for any fixed pair (Q, R), the function T is strictly increasing in π. Therefore, fixing $Q=Q^*(\pi_1)$, $R=R^*(\pi_1)$ and denoting the minimum value of T for $\pi=\pi_i$ by $T^*(\pi_i)$ we get

$T^*(\pi_1)=T(Q^*(\pi_1), R^*(\pi_1), \pi_1) > T(Q^*(\pi_1), R^*(\pi_1), \pi_2)$ for all $\pi_1 > \pi_2$. But $T(Q^*(\pi_1), R^*(\pi_1), \pi_2) > T(Q^*(\pi_2), R^*(\pi_2), \pi_2) = T^*(\pi_2)$, since $Q^*(\pi_2)$, $R^*(\pi_2)$ minimize T for $\pi=\pi_2$. Hence $T^*(\pi_1) > T^*(\pi_2)$ for all $\pi_1 > \pi_2$, i.e., $T^*(\pi) \sim \pi$.

Now note that dividing both sides of (4) by $H/2Q$ and transposing one gets

$$\frac{\pi\mu B}{Q} = \frac{HQ}{2} - \frac{K\mu}{Q}.$$

Therefore, by substituting the above for the last term on the right hand side of (1) we have

$$T^*(\pi) = T(Q^*(\pi), R^*(\pi)) = H \cdot \{Q^*(\pi) + R^*(\pi) - \lambda\}.$$

Since $T^*(\pi) \sim \pi$ and $H > 0$, the result is established.

The implication of the above result is that if π increases then one is sure that the resultant increase in R^* shall offset the decrease in Q^*, if any.

IV. EXAMPLES

The following examples are intended for illustrating the validity of Corollary 2.

EXAMPLE 1: (Normal distribution). From the tables of the probability and loss integrals (see Raiffa and Schlaifer [4] or Wagner [5], page 819) for the standard Normal distribution it is easily verified that $\beta(x)$ is monotone decreasing in x. A condensed table is given below for ready reference.

x	0.5	1.0	1.5	2.5	3.0
$B(x)$	0.1977	0.0833	0.0293	0.0020	0.0003
$\Phi^c(x)$	0.3085	0.1586	0.0668	0.0062	0.0013
$\beta(x)$	0.6408	0.5252	0.4386	0.3226	0.2308

This explains and also establishes the validity of the computational experience of Herron [2] and Kaplan [3] that when the leadtime distribution is Normal, $Q^*(\pi)$ decreases monotonically as π increases.

EXAMPLE 2: (Weibull distribution). For the Weibull distribution defined as

$$\phi(x) = \alpha x^{\alpha-1} \exp(-x^\alpha), \quad \alpha > 0, \; x \geq 0$$

it is known that $\Phi(x) = 1 - \exp(-x^{\alpha})$. For the special case $\alpha = 1/2$, it can be shown by integration that

$$B(x) = 2(1 + x^{1/2}) \exp(-x^{1/2})$$

o that $\beta(x) = 2(1 + x^{1/2}$, which is monotone increasing in x. Therefore, if leadtime demand follows a Weibull distribution with $\alpha = 1/2$, then $Q^*(\pi)$ must increase monotonically with π. (This fact about the Weibull distribution is first noted in [3].)

EXAMPLE 3: (Pareto distribution.) Consider the Pareto distribution

$$\phi(x) = (\alpha/\beta)(\beta/x)^{\alpha+1}, \quad \alpha \geq 3, \quad x \geq \beta > 0.$$

The related distribution function is given by $\Phi(x) = 1 - (\beta/x)^{\alpha}$.
t follows by integration that

$$B(x) = (\beta^{\alpha} x^{1-\alpha} - \alpha\beta)/(\alpha - 1).$$

Hence

$$\beta(x) = (x - \alpha\beta^{1-\alpha} x^{\alpha})/(\alpha - 1)$$

which is monotone decreasing in x for $\alpha \geq 3$ and $x \geq \beta > 0$.
Therefore, like the Normal distribution, the Pareto distribution for leadtime demand has the property hat as π is increased $Q^*(\pi)$ must decrease monotonically.

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A MEAN COST APPROXIMATION FOR TRANSPORTATION PROBLEMS WITH STOCHASTIC DEMAND

Dan Wilson*

Arizona Health Planning Authority
Phoenix, Arizona

ABSTRACT

Among the many tools of the operations researcher is the transportation algorithm which has been used to solve a variety of problems ranging from shipping plans to plant location. An important variation of the basic transportation problem is the transportation problem with stochastic demand or stochastic supply. This paper presents a simple approximation technique which may be used as a starting solution for algorithms that determine exact solutions. The paper indicates that the approximation technique offered here is superior to a starting solution obtained by substituting expected demand for the random variables.

INTRODUCTION

Over the years, the transportation model has been one of the work horses of operations research and has been applied to a variety of problems. Because of its extensive use and the number of problems that can be modeled as transportation problems, the stochastic versions of the transportation model become important and necessary extensions of the basic tool.

In this paper, a technique is presented for obtaining approximate solutions to transportation problems with stochastic demands and penalty costs. The approximate solution may be used as a starting solution for more sophisticated algorithms which yield exact solutions. The technique reduces the stochastic, nonlinear and often all integer problem to an approximate linear deterministic equivalent that can be solved as a standard transportation problem. An attractive feature of the approach is its ability to retain the original size of the cost matrix. This is in contrast to other approaches that have been developed to solve the problem.

No discussion will be offered here on other work that has been done on the problem at hand. Instead, interested readers are directed to the references of this paper for techniques that find either approximate or exact solutions to the problem. It should be mentioned that while this paper discusses the problem from the stochastic demand viewpoint, the development applies also to problems with stochastic supply and deterministic demand.

THE STOCHASTIC DEMAND FORMULATION

The transportation problem with stochastic demand, deterministic supply and penalty costs for overages and shortages is given as follows:

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*Current affiliation: Samaritan Health Service, Phoenix, Arizona.

Minimize

$$Z=\sum_i \sum_j c_{ij}x_{ij}+\sum_j E(P_j) \tag{1}$$

subject to

$$\sum_j x_{ij} \leq a_i \tag{2}$$

$$x_{ij} \geq 0 \qquad \text{for all } i \text{ and } j,$$

where

$$b_j=\sum_i x_{ij}. \tag{3}$$

$$E(P_j)=h_j \int_0^{b_j} (b_j-D_j)f(D_j)\,dD_j+s_j \int_{b_j}^{\infty} (D_j-b_j)f(D_j)\,dD_j \tag{4a}$$

in the continuous case and where for the discrete case,

$$E(P_j)=h_j \sum_{D_j=0}^{b_j} (b_j-D_j)p(D_j)+s_j \sum_{D_j=b_j+1}^{\infty} (D_j-b_j)p(D_j) \tag{4b}$$

and where

- c_{ij} denotes the cost of shipping one unit from i to j,
- x_{ij} denotes the quantity of items shipped from i to j,
- h_j denotes the cost per unit of excess inventory at j,
- s_j denotes the penalty cost per unit of inventory shortage at j,
- D_j denotes the random variable for demand at j,
- b_j is a deterministic point estimate of D_j,
- $f(D_j)$ denotes the probability density function of demand for the continuous case, and
- $p(D_j)$ denotes the probability mass function of demand at j in the discrete case.

UPPER AND LOWER BOUNDS ON THE MEAN SHIPPING COST

We begin the development by noting that the total expected costs to supply b_j units to destination j is the sum of the shipping costs to j and the expected overage and shortage costs at j. Mathematically this is written as

(5)

The average shipping cost to j is then

$$\bar{c}_j = \sum_i c_{ij}x_{ij} \Big/ \sum_i x_{ij}. \tag{6}$$

According to (3) we can replace the denominator of (6) with b_j to arrive at

$$\bar{c}_j = \sum_i c_{ij}x_{ij} \Big/ b_j, \tag{7}$$

so that the total shipping cost into destination j is just

$$\sum_i c_{ij}x_{ij} = \bar{c}_j b_j. \tag{8}$$

Substituting (8) into (5), we can rewrite the expected costs due to j as

$$z_j = \bar{c}_j b_j + E(P_j). \tag{9}$$

Summing over all j leads to a reformulated objective function which we substitute into the problem formulation for (1).

$$Z = \sum_j \left[\bar{c}_j b_j + E(P_j)\right]. \tag{10}$$

In this form, the objective function may be recognized as the summation of n single period stochastic inventory problems (newsboy problems) with setup costs. Further, if the row constraints given by (2) are temporarily ignored, then the demand points become independent and (10) is separable. With this in mind, we take the partial derivative of (10) with respect to b_j in an attempt to minimize Z. For a single demand point, this leads to the well-known result for the continuous case of

$$\delta Z/\delta b_j = (s_j + h_j)F(b_j) + \bar{c}_j - s_j = 0 \tag{11a}$$

and

$$F(b_j) = (s_j - \bar{c}_j)/(s_j + h_j), \tag{11b}$$

where $F(b_j)$ is the cumulative probability distribution for demand at point j. The discrete analogue is given by

$$F(b_j - 1) \leq (s_j - \bar{c}_j)/(s_j + h_j) \leq F(b_j). \tag{11c}$$

With an *a priori* knowledge of s_j, h_j, and $\bar{c}_j$ we could compute the critical ratio $(s_j - \bar{c}_j)/(s_j + h_j)$, and by referring to the cumulative probability distribution or the cumulative probability mass func-

tion, depending on the case, we could determine a value for b_j. Once determined, b_j could then serve as a point estimate for D_j, be treated as deterministic, and we could solve a standard transportation problem thereby obtaining values for the x_{ij} and introducing the previously ignored supply constraints given by (2). Unfortunately, while s_j and h_j are generally known beforehand, the same cannot be said for $\bar{c}_j$ since it is a function of the allocations, x_{ij}, and cannot be known until after the allocations have been made. We do know, however, that

$$\min_i c_{ij} \leq \bar{c}_j \leq \max_i c_{ij}, \tag{12}$$

therefore,

$$[(s_j - \max_i c_{ij})/(s_j + h_j)] \leq (s_j - \bar{c}_j)/(s_j + h_j) \leq [(s_j - \min_i c_{ij})/(s_j + h_j)] \tag{13}$$

so that we have determined upper and lower bounds on both the mean shipping cost and the critical ratio. It has been shown elsewhere [4] that (13) can be used to determine upper and lower bounds on b_j and that an *a priori* bounded model can be formulated to arrive at reasonably good approximate solutions, the "goodness" depending upon the amount of variability in the c_{ij}. A disadvantage of the *a priori* bounded model is that it results in a capacitated transportation problem and requires either an expanded matrix using the standard transportation algorithm, or requires an algorithm such as the primal-dual which can handle capacitated transportation problems directly. Here, we remove the capacity restrictions present in the *a priori* bounded model.

An important conclusion from development of the upper and lower bounds on $\bar{c}_j$ is that any point estimate of $\bar{c}_j$ must lie between the upper and lower bounds given in (12) in order to be feasible. Further, any value of b_j selected as a point estimate for D_j, must be derived from a feasible estimate for $\bar{c}_j$. In particular, the common technique of replacing D_j with its expected value as a point estimate may not result in a feasible solution. From personal experience of the author, some analysts in practice when an approximate solution is acceptable, will often use the expected value substitution and rely on the results as being a reasonable approximation. This can be checked, however. For example, if we let $b_j = E(D_j)$ in (11b) and solve for $\bar{c}_j$ we arrive at

$$\bar{c}_j = s_j - (s_j + h_j)F(E[D_j]) \tag{14}$$

and by (12), $\bar{c}_j$ must lie between the upper and lower bounds to be feasible. It is clear that substituting the expected value for b_j does not guarantee this in the continuous case. A similar approach for the discrete case may be used to illustrate the situation there. Because of the necessity to choose a feasible value for $\bar{c}_j$ and a resulting feasible value for b_j, the approach described here is deemed superior to the expected value approximation method.

ESTIMATING THE MEAN SHIPPING COST

On reflection, one could choose any value within the upper and lower bound limits as a feasible estimate of $\bar{c}_j$, however, it would be advantageous to select a value that recognizes the behavior of the allocation process in the transportation algorithm. Since the algorithm attempts to minimize shipping costs, we would expect the *a posteriori* value of $\bar{c}_j$ to be closer to the smallest c_{ij} in the column than the

largest. Now consider the following normalized shipping cost.

$$q_{ij} = c_{ij}/c_{\cdot j}, \tag{15}$$

where

$$c_{\cdot j} = \sum_i c_{ij}.$$

The normalized shipping cost is viewed as an estimate of the likelihood that an allocation will *not* be made to cell i, j. The greater the value of c_{ij}, the less likely that it will receive an allocation. Now consider an estimate of the likelihood that an allocation *will* be made to cell i, j.

$$p_{ij} = (1-q_{ij}) \Big/ \sum_i (1-q_{ij}) \tag{16}$$

where the summation in the denominator serves to normalize the values over the column and excludes any cells that have been denoted as infeasible allocations and excludes any slack variables. Using the p_{ij} as weights, we determine estimates of the anticipated allocations for the various cells by writing

$$\hat{x}_{ij} = p_{ij} b_j \tag{17}$$

Substituting (17) into (7), we get

$$\tilde{c}_j \approx \hat{c}_j = \sum_i c_{ij}\hat{x}_{ij}/b_j$$

$$= \sum_i c_{ij}(p_{ij}b_j)/b_j$$

$$\hat{c}_j = \sum_i c_{ij}p_{ij}. \tag{18}$$

Finally, substituting (15) into (16) and (16) into (18) we arrive at the final form for estimating the mean shipping cost.

$$\hat{c}_j = \left(\left[\sum_i c_{ij}\right]^2 - \sum_i c_{ij}^2\right) \Big/ (m-1)\left[\sum_i c_{ij}\right]. \tag{19}$$

Where m is the total number of cells in the column less the number of cells denoted as infeasible allocations (infinite cost) and less the number of slack variables (dummies) that may be in the column.

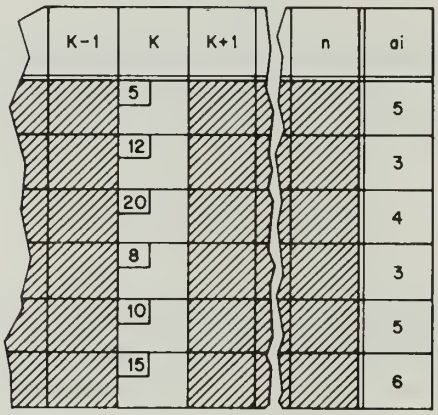
AN EXAMPLE USING THE DISCRETE CASE

To illustrate the procedure just discussed, consider the kth column in a cost matrix for a transportation problem with discrete stochastic demand. Suppose that demand varies between 10 and 19

units, all of which are equally likely so that the probability mass function is described by

$$p(D_k) = 0.1 \qquad \text{for } 10 \leq D_k \leq 19. \tag{20}$$

We let the per unit overage cost be $h_k = 10$ and the per unit shortage cost be $s_k = 25$. Next, suppose we have costs c_{ik} and supplies a_i as shown in Figure 1.



A partial cost matrix.

Scanning the kth column, we note that the largest c_{ik} is $c_{3k} = 20$ while the smallest is $c_{1k} = 5$. From (12) we know that the value selected to estimate $\bar{c}_k$ must lie between 5 and 20 in order for the resulting estimate of demand, D_k to be feasible. Applying (19), we arrive at

$$\hat{c}_k = \frac{(5+12+20+8+10+15)^2 - (5^2+12^2+20^2+8^2+10^2+15^2)}{(6-1)\quad(5+12+20+8+10+15)}$$

$$\hat{c}_k = 11.2. \tag{21}$$

The result of 11.2 contrasts to the midpoint of 12.5 and illustrates the tendency of the method to bias the estimate toward the smaller valued c_{ij} thus recognizing the minimization behavior of the transportation algorithm. With this estimate of the mean cost for column k, one would substitute the estimate into (11c) for the discrete case and refer to the cumulative probability mass function of demand at point k to arrive at b_k, the point estimate for D_k. The probability mass function for the example is defined in (20), and the critical ratio from (11c) becomes

$$F(b_k - 1) \leq (25 - 11.2)/(25 + 10) \leq F(b_k)$$

$$F(b_k - 1) \leq 0.395 \leq F(b_k). \tag{22}$$

From (20) we can determine that $F(12) = P(D_k \leq 12) = 0.3$ and that $F(13) = P(D_k \leq 13) = 0.4$ so that $b_k = 13$ according to (22). This compares to the expected value of demand, $E(D_k) = 14.5$. With this estimate of demand, the analyst may proceed to treat the estimate as deterministic and solve a Standard transportation problem.

CONCLUSIONS

In many real life situations one is willing to accept suboptimization in favor of ease of calculation, especially when available expertise is not acquainted with methods to determine exact solutions to stochastic transportation problems. For this reason, analysts in practice often substitute the expected value of demand into the standard formulation of the transportation problem, solve it and rely on the results as being feasible although suboptimal. We have shown here that the expected value substitution approach may not necessarily yield a feasible solution to the stochastic formulation, and have presented an approach to arrive at an estimate that would be feasible and still allow the analyst who is willing to accept suboptimization to solve a standard transportation problem as a deterministic approximation to the stochastic transportation problem.

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ON A SUPPLEMENTARY TIME VARIABLE APPROACH TO BATTLE ANALYSIS

Kare M. Mjelde

SHAPE Technical Center
The Hague, Netherlands

ABSTRACT

A supplementary time variable approach is used to analyze a battle where attackers search for defended logistical targets and the future actions of an attacker depend on its activities in the whole time interval since arrival at the operating area. As compared with the method discussed in this paper a corresponding Markov chain analysis of the problem requires a considerably greater number of mathematical relations for state transition probabilities with less possibilities in obtaining numerical solutions effectively and with less parametrical insight. A supplementary time variable is introduced to describe the arrival instants of attackers at the operating area, and is then used to express the probability generating functions of the losses at an arbitrarily selected time instant in terms of a general description of the capabilities of an individual attacker in the time interval since arrival. These capabilities are then related to a selected set of lower level parameters describing a specific battle situation by using a small set of integral relations of the multiple convolution type with solutions expressed as Laplace transforms.

INTRODUCTION

The situation examined in this paper is a military one where attackers search for defended targets, and the future actions of an attacker depend on its activities in the whole time interval since arrival at the operating area. There are analogous situations in other fields of investigation—for example, the effects of a given quantity of a polluting chemical depend on the time since insertion into the environment, and the growth of an urban activity depends on the time since it was established in the district.

The battle is described in terms of submarines attacking defended convoys (although the discussion would be equally relevant if the attackers were aircraft or the targets were naval forces) and is analyzed in terms of the losses of units of both sides. The probability generating functions of the total losses at an arbitrarily selected time instant are obtained by the introduction of a supplementary time variable describing the arrival instants of the attackers at the operating area. The total losses are expressed in terms of the following three quantities, which give a general description of the performance of any individual attacker in the time interval since arrival:

1. The probability that the attacker is destroyed.
2. A random variable which is equal to the number of times the attacker releases weapons against a target.
3. A random variable or a probability describing the level of destruction inflicted on a target when an attacker releases weapons.

The arrival instants of attackers are described in two different ways: by a Poisson stream with a

time dependent rate or by regular arrivals at given time instants; the supplementary time variable provides a link between the stream of arriving attackers and the activities of a single attacker in a time interval since arrival. This approach has been introduced in Reference [4] for the case where attackers arrive according to a Poisson stream with constant rate. The method differs from the Markov-Lanchester theory in that it is not based on transitions between states depending on the total number of attackers and defenders in the system.

In this paper a specific battle situation is used to demonstrate how the introduction of a supplementary time variable enables the battle to be analyzed quantitatively by a small number of mathematical relations, as compared with a large and computationally ineffective set of differential-difference equations for state transition probabilities associated with the Markov chain approach. The method employed for the demonstration is to express the quantities 1, 2, and 3 above in terms of a selected set of lower-level parameters describing the specific situation under consideration.

THE SITUATION

Defended merchant ship convoys are attacked by submarines operating independently of each other within an operating area. When a submarine detects a convoy inside the region of approach, Koopman [3], an encounter is said to occur. The convoy defences have some probability of destroying an approaching submarine before it reaches an attack position; the time that elapses between an encounter and the corresponding attempt to penetrate the convoy defences to reach an attack position is a random variable, the approach time. An attack consists of a sequence of weapon expenditures resulting in the destruction of a random number of ships. A submarine must escape through the convoy defences after an attack, with some probability of being destroyed. There is a random recovery time between a successful escape through the convoy defences after an attack, until the instant when the submarine again starts searching for convoys. A submarine's actions are limited by the maximum number of attacks it can perform, as determined by the total number of weapons on board, and by its endurance, which controls the maximum time it can stay in the operating area.

A MARKOV CHAIN APPROACH

In the analysis it is assumed that a submarine in the operating area must be in one of five states depending on whether it:

1. is searching for convoys,
2. is approaching a convoy to reach an attack position from which weapons can be released,
3. is recovering after a successful attack,
4. is leaving the operating area, or
5. has been destroyed by the defences of a convoy.

Since a submarine leaves the operating area because of weapon or endurance limitation, each of these five cases is further subdivided according to the number of attacks a submarine has made and the number of time intervals, of some selected size, the submarine has been in the operating area.

A state vector, giving the number of attackers in each state, can be used to describe the system; the number of possible state vectors are greater than the product of the number of states and the number of attackers than can be in the operating area simultaneously. The Markov chain description must

take account of the fact that transitions between state vectors depend on the search, approach, and recovery time distributions corresponding to the individual submarines. These can, in principle, either be described as sojourn times or introduced in the state vector by the subdivision of the times considered into smaller intervals. In any case, a large number of mathematical relations for transition probabilities between state vectors is required; this number is equal to the number of state vectors squared, and values of several hundreds or even thousands appear in many cases.

An alternative proposal is the use of the supplementary time variable model, which is defined below in terms general functional equations. The application of this model to the analysis of a specific battle situation is described.

THE SUPPLEMENTARY TIME VARIABLE APPROACH

The total number of merchant ships and submarines that are destroyed before an arbitrarily selected time instant t is determined by introducing a supplementary time variable θ describing the arrival instant of a submarine at the operating area. The attack capability of a submarine arriving at time θ is defined by the following input:

$u(\theta, t)$: The probability that the submarine is destroyed before time t, where $t \geqslant \theta$.

$N(\theta, t)$: A random variable which is equal to the number of convoy attacks the submarine performs in a time interval $[\theta, t]$.

K: A random variable describing the number of ships destroyed when a submarine releases weapons after a successful penetration of the convoy defences.

The following probability distribution and generating functions are defined:

$$V_n(\theta, t) = P(N(\theta, t) = n).$$

$$G(\theta, t, z) = \sum_{n=0}^{\infty} V_n(\theta, t) z^n.$$

$$F(z) = \sum_{n=0}^{\infty} P(K = n) z^n.$$

The output of the model is defined by the probability generating functions $M(t, z)$, $H(t, z)$, and $S(t, z)$ of the following random variables:

$M(t, z)$: The total number of attacks made by all submarines before time t.

$H(t, z)$: The number of merchant ships destroyed before time t.

$S(t, z)$: The number of submarines destroyed before time t.

If the submarines arrive at the operating area according to a Poisson stream with time dependent rate $\mu(\theta) = 0$ for $\theta \leqslant 0$, it follows from the arguments of Reference [4] that:

$$M(t, z) = \exp\left\{ \int_0^t \mu(\theta) [G(\theta, t, z) - 1] d\theta \right\}, \tag{1}$$

$$S(t, z) = \exp\left\{ \int_0^t \mu(\theta)\, u(\theta, t)\, d\theta\, (z-1) \right\}, \tag{2}$$

and, by the theory of branching processes [2] that:

$$H(t, z) = M(t, F(z)). \tag{3}$$

If arrivals occur regularly at given time instants θ_k, where $k = 1, 2, \ldots$, then for $\theta_n \leq t < \theta_{n+1}$:

$$M(t, z) = \prod_{k=1}^{n} G(\theta_k, t, z), \tag{4}$$

$$S(t, z) = \prod_{k=1}^{n} [1 + (z-1)\, u(\theta_k, t)]. \tag{5}$$

Regular arrivals can be superimposed on a Poisson arrival process by multiplying the corresponding generating functions for the losses. However, for analytical purposes it is more convenient to use the exponential expressions given in (1) and (2) rather than the product formulas (4) and (5). Furthermore, factors such as navigation errors, random speed variations and delays in the home port tend to make the arrivals random.

ANALYSIS OF A SPECIFIC SITUATION

In the analysis of a specific battle situation the probabilities $u(\theta, t)$ and $V_n(\theta, t)$ associated with the supplementary time variable model must be expressed by parameters describing the performance of a submarine in the situation analysed. Then (1) and (2), or (4) and (5), are used together with (3) to obtain the total losses of ships and submarines before any time t.

This procedure is now applied to a specific battle situation that is described by introducing the following additional parameters and assumptions.

A single submarine encounters convoys at a rate λ, which corresponds to a Poisson process. The endurance limited time a submarine can stay in the operating area is denoted by T and is determined by geographical distances, the amount of fuel on board, and similar factors. The maximum number of attacks a submarine can perform before all weapons have been used is an integer, R. The parameters P_1 and P_2 give the probabilities that a submarine is not destroyed (1) by the defences of a convoy during penetration to reach an attack position, and (2) during escape after weapons release. Finally, the cumulative distribution functions of the random approach and recovery times corresponding to an attack are designated by $A_1(v)$ and $A_2(v)$ respectively; their convolution, which is the distribution function of the total duration of an attack, is denoted by $A(v)$.

A submarine which arrived at the operating area at time θ is considered at a time $(\theta + t)$, where t is the time that has elapsed since the submarine arrived. Since the parameters λ, P_1, and P_2 are assumed to be constants, the probabilities $u(\theta, \theta + t)$ and $V_n(\theta, \theta + t)$ depend only on the time t, and can be written as a function of a single time variable; $u(t) = u(\theta, \theta + t)$ and $V_n(t) = V_n(\theta, \theta + t)$.

It is convenient to introduce the probability $b_n(t)$ that a submarine which arrived at the operating area at a time θ has made n attacks before the time $(\theta + t)$ and is searching for convoys at that time.

First these probabilities are expressed in terms of the previously defined inputs and then used to determine the probabilities $u(t)$ and $V_n(t)$, $n = 1, \ldots, R$.

Then for $t \geq T$

$$b_n(t) = 0 \qquad n \geq 0, \tag{6}$$

and for $0 \leq t < T$.

$$b_0(t) = \exp(-\lambda t). \tag{7}$$

$$b_n(t) = \int_{x=0}^{x=t} \int_{\xi=x}^{\xi=t} b_{n-1}(x)\lambda dx P_1 P_2 dA(\xi - x)\, b_0(t-\xi) \qquad 1 \leq n < R. \tag{8}$$

$$b_n(t) = 0 \qquad n \geq R. \tag{9}$$

The explanations are as follows:

Equation (7) gives the probability that a submarine encounters no convoys during a time t in the operating area and is justified by noting that if an encounter is made at a time x then one of the following situations results at time $t > x$:

(1) The submarine is in the approach phase of an attack

(2) It has been destroyed

(3) It has made at least one attack.

To show that (8) is true demands that the submarine makes its n'th encounter at a time x after arrival at the operating area. Then P_1P_2 is the probability of the submarine surviving the attack, and $dA(\xi - x)$ is the probability that the submarine again starts searching for convoys in the time interval $[\xi, \xi + d\xi]$. Finally $b_0(t-\xi)$ is the probability of no additional encounter in the time interval $[\xi, t]$. Integration over all possible values of x and ξ gives (8).

To solve Equations (6) to (9), it is useful to introduce functions $b_n'(t)$ that are determined by Equations (7) to (9) for all $t \geq 0$, neglecting the restriction $t < T$. Then (6) and the fact that $b_n(t) = b_n'(t)$ for $0 \leq t < T$ determine $b_n(t)$ for all $t \geq 0$. Noting that (8) has the form of a double convolution, taking Laplace transforms give

$$L(b_0') = \frac{1}{s+\lambda}$$

$$L(b_n') = (\lambda P_1 P_2)^n \frac{(\phi_1(s)\phi_2(s))^n}{(s+\lambda)^{n+1}} \qquad 1 \leq n < R$$

$$L(b_n') = 0 \qquad n \geq R,$$

where

$$L(b_n') = \int_0^\infty b_n'(t) \exp(-st)\, dt$$

and

$$\phi_i(s) = \int_0^\infty \exp(-st)\, dA_i(t) \qquad i = 1, 2.$$

The last integral is evaluated with respect to the Stieltjes measure generated by the distribution function $A_i(t)$.

For $0 \leq t \leq T$ the sum

$$b(t) = \sum_{n=0}^{R-1} b_n(t)$$

gives the probability that a submarine is searching for convoys at time t after its arrival at the operating area. Since a submarine may be in any other of the five states that have previously been defined, this sum is usually smaller than one. Note that a submarine leaves the operating area before time T, if all weapons have been used.

The probabilities $V_n(t)$ are given for $t > T$ by

$$V_n(t) = V_n(T) \tag{10}$$

and for $0 \leq t \leq T$:

$$V_0(t) = \exp(-\lambda t) + \int_{x=0}^{x=t} \lambda \exp(-\lambda x)\, dx \left\{ \int_{\xi_1=x}^{\xi_1=t} dA_1(\xi_1 - x)(1 - P_1) + \int_{\xi_1=t}^{\xi_1=\infty} dA_1(\xi_1 - x) \right\}, \tag{11}$$

$$V_n(t) = \int_{x=0}^{x=t} b_{n-1}(x)\, \lambda dx \int_{\xi_1=x}^{\xi_1=t} dA_1(\xi_1 - x) \left\{ P_1(1 - P_2) + P_1 P_2 \left[\int_{\xi_2=t}^{\infty} dA_2(\xi_2 - \xi_1) + \int_{\xi_2=\xi_1}^{\xi_2=t} dA_2(\xi_2 - \xi_1) V_0(t - \xi_2) \right] \right\}, \qquad 1 \leq n \leq R \tag{12}$$

$$V_n(t) = 0 \qquad n > R. \tag{13}$$

The equation (11) for $V_0(t)$ says that either there is no encounters during the time t or, if there is an encounter at a time $x < t$, the submarine is either destroyed during penetration of the convoy defences before time t or the approach phase of the attack is not finished before time t. The arguments leading to Equation (12) are similar. The n'th encounter must occur at a time $x < t$ and a successful penetration of the defences at a time ξ_1 such that $x \leq \xi_1 \leq t$. The submarine is either destroyed while leaving the convoy or, if not, it either finishes its recovery phase at a time $\xi_2 > t$ or $\xi_1 < \xi_2 < t$ and there are no additional attacks in the time interval $[\xi_2, t]$.

It is convenient to define $V_n'(t)$ by the Equations (11) and (12) without the restriction $t \leq T$, recording that $V_n(t) = V_n'(t)$ for $0 \leq t \leq T$ and noting (10). Then, from the convolution property of Laplace transforms it follows that:

$$L(V_0') = \frac{1}{s} - \frac{\lambda P_1 \phi_1(s)}{s(s+\lambda)}$$

$$L(V_n') = \frac{\lambda P_1 \phi_1(s)}{s} [L(b_{n-1}') - L(b_n')] \qquad 1 \leq n \leq R$$

$$L(V_n') = 0 \qquad R < n,$$

where the $L(b_n')$ have already been found, and can be inserted in the above equations.

As a check, note that

$$\sum_{n=o}^{R} L(V_n') = 1/s,$$

which implies that the sum

$$\sum_{n=o}^{R} V_n(t) = 1,$$

as it should.

Now, by similar arguments, the probability $u(t)$ is given by $u(t) = u'(t)$ for $0 \leq t \leq T$ and $u(t) = u(T)$ for $t \geq T$, where $u'(t)$ is given by the Laplace transform:

$$L(u') = \lambda(1 - P_1 P_2) \frac{\phi_1(s)}{s} \sum_{n=0}^{R-1} L(b_n').$$

The total losses can now be determined, from the probabilities $V_n(t)$ and $u(t)$, by using Equations (1) and (2) or (4) and (5). To determine the losses by the supplementary time variable method there are three equations for the generating functions $M(t, z)$, $S(t, z)$, and $H(t, z)$ in terms of the probabilities $V_n(t)$, $0 \leq n \leq R$, and $u(t)$; these probabilities are governed by $(R+2)$ relations for the Laplace transforms of the associated probabilities $V_n'(t)$ and $u'(t)$. The latter probabilities can either be evaluated explicitly, for example, if the approach and recovery phases of an attack assume specific values with probability 1, or numerically by applying a fast Laplace transform inversion procedure, see [1] and [5]. The probability distributions of the losses can then be obtained from the generating functions, either explicitly or by the substitution of $z = \exp(-s)$ and Laplace inversion, which is a very fast operation and makes knowledge of the generating function almost equivalent to knowledge of the distribution itself, at least numerically.

CONCLUDING REMARKS

The supplementary time variable approach discussed in this paper is applicable to a wide class of situations, where it is more effective in giving numerical solutions and parametrical insight than a corresponding Markov chain model. This is particularly true if a Markov chain analysis requires definition of state vectors with a large number of components and stochastic sojourn times that are not im-

mediately obtainable from the parameters defining the problem, and necessitating the use of possibly complex submodels.

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APPLICATION OF THE FUNDAMENTAL THEOREM OF GAMES TO AN EXAMPLE CONCERNING ANTIBALLISTIC MISSILE DEFENSE

John S. Croucher

Macquarie University
North Ryde, New South Wales
Australia

ABSTRACT

This paper considers a two sided resource allocation game in which both players initially have fixed resources which may be distributed over various targets. Their effectiveness depends on the manner of distribution and also on the strategy of the opponent, a natural payoff function for such a situation being used. The complete solution to the game is derived and a numerical example given.

I. INTRODUCTION

The problem examined is one where a number of installations (targets) are attacked and defended with the use of missiles. The variables in the game are assumed to be continuous, and the approach handles many situations apart from the one considered here.

Suppose that an attacker has k ballistic missiles (BM's) with which to attack k targets, and that there are X separate reentry vehicles (RV's) armed with nuclear warheads which may be distributed among the BM's. It is assumed that exactly one BM is used per target, and that any BM can carry an arbitrary number of warheads. The defender has Y long-range antiballistic missiles (ABM's) to defend the k targets. If successful, an ABM weapon destroys an attacking missile and all of the reentry vehicles it carries.

When an attacking BM approaches the ABM defense, the RV's are sufficiently close together such that if the BM is hit by an ABM, all of its RV's are destroyed. However, if the BM penetrates this defense, the RV's separate in such a way that each becomes a weapon, any one of which is capable of destroying the target. In this case the target is protected only by a terminal defensive system which consists of an arbitrarily large number of short range weapons. The terminal defense then attempts to destroy each of the RV's individually. (As an alternative to a terminal defense, the terminal attrition of RV's can be attributed to lack of RV reliability and accuracy.)

II. FORMULATION

Suppose that a target is attacked by a BM carrying x RV's and is defended by y ABM's. A natural probability function, $p(x, y)$, representing the probability that the attacker destroys the target and thus gains its full value is

$$p(x, y) = [1 - \exp(-ax)] \exp(-by).$$

Where "a" and "b" may be considered as vulnerability factors associated with the target. The function p has the properties

$$\text{(i)} \quad \lim_{x \to 0} p(x, y) = 0 \qquad \text{for fixed } y,$$

$$\text{(ii)} \quad \lim_{y \to \infty} p(x, y) = 0 \qquad \text{for fixed } x.$$

Suppose that the k targets are labelled, such that the ith target has the value v_i and vulnerability factors b_i and a_i. If the attacking BM carries x_i RV's and the defender has y_i AMB's, the total (expected) payoff to the attacker is

$$F(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{k} v_i p(x_i, y_i) = \sum_{i=1}^{k} v_i [1 - \exp(-b_i x_i)] \exp(-a_i y_i), \tag{1}$$

where

$$\mathbf{x} = (x_1, \ldots, x_k), \quad \mathbf{y} = (y_1, \ldots, y_k)$$

are the strategies chosen by the attacker and defender, respectively, subject to the conditions

$$\sum_{i=1}^{k} x_i = X, \qquad \sum_{i=1}^{k} y_i = Y.$$

It is assumed that the values gained at each target are additive and that both sides allocate their entire resources.

III. SOLUTION

The solution will be found by using the fundamental theorem of games in the following form; its proof can be found in Kakutani [3].

THEOREM 1: Suppose $F(\mathbf{x}, \mathbf{y})$ is a continuous function on a space $K \times L$, where K and L are compact and convex. Suppose that the set of points $X(\mathbf{y})$ yielding the maximum to $F(\mathbf{x}, \mathbf{y})$ for $\mathbf{y}$ fixed is convex for each such $\mathbf{y}$, and that the set of points $Y(\mathbf{x})$ yielding the minimum to $F(\mathbf{x}, \mathbf{y})$ for $\mathbf{x}$ fixed is convex for each such $\mathbf{x}$. Then the game has a pure strategy solution.

Theorem 1 will be used in conjunction with Theorem 2 which concerns a particular case of the function F. The proof of Theorem 2 follows directly from the definitions of convexity and concavity.

THEOREM 2: Suppose

$$F(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{k} f_i(x_i, y_i)$$

where $f_i(x_i, y_i)$ is concave in x_i for fixed y_i and convex in y_i for fixed x_i. Let $X(\mathbf{y})$ be the set of points which maximizes $F(\mathbf{x}, \mathbf{y})$ for fixed $\mathbf{y}$, and $Y(\mathbf{x})$ be the set of points which minimizes $F(\mathbf{x}, \mathbf{y})$ for fixed

$\mathbf{x}$. Then the sets $X(\mathbf{y})$ and $Y(\mathbf{x})$ are both convex.

Let

$$f_i(x_i, y_i) = v_i[1 - \exp(-b_i x_i)] \exp(-a_i y_i).$$

It can be easily shown that $f_i(x_i, y_i)$ are concave in x_i for fixed y_i, and convex in y_i for fixed x_i. Combining Theorems 1 and 2, a pure strategy solution exists for the game with the payoff function F defined by (1). Then there exists a pair of optimal strategies $(\mathbf{x}^0, \mathbf{y}^0)$ where

$$\mathbf{x}^0 = (x_1^0, \ldots, x_k^0), \qquad \mathbf{y}^0 = (y_1^0, \ldots, y_k^0)$$

such that

$$F(\mathbf{x}^0, \mathbf{y}) \geq F(\mathbf{x}^0, \mathbf{y}^0) \qquad \text{for all } \mathbf{y}$$

and

$$F(\mathbf{x}, \mathbf{y}^0) \leq F(\mathbf{x}^0, \mathbf{y}^0) \qquad \text{for all } \mathbf{x}.$$

In other words, $\mathbf{x}^0$ maximizes the expression

$$\sum_{i=1}^{k} v_i[1 - \exp(b_i x_i)] \exp(-a_i y_i^0) \tag{2}$$

and $\mathbf{y}^0$ minimizes the expression

$$\sum_{i=1}^{k} v_i[1 - \exp(-b_i x_i^0)] \exp(-a_i y_i). \tag{3}$$

The min–max and max–min solutions can be found from a version of the Neyman-Pearson lemma which is sometimes referred to as Gibbs Lemma. Its proof can be found in Danskin [2].

THEOREM 3 (Gibbs Lemma): Suppose that $\mathbf{x}^0 = (x_1^0, \ldots, x_k^0)$ maximizes $\sum_i f_i(x_i)$ subject to the conditions

$$x_i > 0, \qquad \sum_i x_i = X.$$

If the f_i are differentiable, then there exists a λ such that:

$$\frac{\partial f_i(x_i^0)}{\partial x_i} = \lambda \qquad \text{if } x_i^0 > 0$$

$$\leq \lambda \qquad \text{if } x_i^0 = 0.$$

Applying Theorem 3 to (3) (and making a change of sign), there exists a μ such that:

$$v_i a_i [1 - \exp(-b_i x_i^0)] \exp(-a_i y_i^0) = \mu \qquad \text{if } y_i^0 > 0 \tag{4}$$

$$\leqslant \mu \qquad \text{if } y_i^0 = 0.$$

Similarly from (2) there exists a λ such that

$$v_i b_i \exp[-(b_i x_i^0 + a_i y_i^0)] = \lambda \qquad \text{if } x_i^0 > 0. \tag{5}$$

$$\leqslant \lambda \qquad \text{if } x_i^0 = 0.$$

The solution will be derived using a series of lemmas. Full details of all the proofs, which may be found in Croucher [1], are left to the interested reader.

LEMMA 1: $\mu > 0$ and $\lambda > 0$.

PROOF: Since $X > 0$, for some i, $x_i^0 > 0$. The left side of (4) is positive for that value of i, and hence $\mu > 0$. That $\lambda > 0$ follows immediately from (5).

LEMMA 2: $y_i^0 > 0$ implies $x_i > 0$.

PROOF: if $y_i^0 > 0$, then the top line of (4) holds. Since

$$\mu > 0, \qquad \text{then } x_i^0 > 0.$$

LEMMA 3: $x_i^0 > 0$ if and only if $v_i b_i > \lambda$.

LEMMA 4: if $x_i^0 > 0$ and $y_i^0 > 0$, then

$$x_i^0 = (1/b_i) \ln (1 + \mu b_i / \lambda a_i),$$

$$y_i^0 = (1/a_i) \ln (v_i / (\mu/a_i + \lambda/b_i)).$$

LEMMA 5: if $x_i^0 > 0$ and $y_i^0 = 0$, then

$$x_i^0 = (1/b_i) \ln (v_i b_i / \lambda).$$

LEMMA 6: $x_i^0 > 0$ and $y_i^0 > 0$ if and only if $v_i > \mu/a_i + \lambda/b_i$.

PROOF: Suppose $v_i > \mu/a_i + \lambda/b_i$. Then $v_i b_i > \lambda$, which from Lemma 3 implies $x_i > 0$. Assume $y_i^0 = 0$. Then the top line of (5) and the bottom line of (4) combine to yield $v_i \leqslant \mu/a_i + \lambda/b_i$, which is a contradiction. Hence $y_i^0 > 0$.

Suppose $x_i^0 > 0$ and $y_i^0 > 0$. The top line of (5) and Lemma 4 combine to yield $v_i > \mu/a_i + \lambda/b_i$.

LEMMA 7: (i) $x_i^0 = y_i^0 = 0$ if and only if $0 < v_i < \lambda/b_i$.

(ii) $x_i^0 > 0$ and $y_i^0 = 0$ if and only if $\lambda/b_i < v_i < \mu/a_i + \lambda/b_i$.

(iii) $x_i^0 > 0$ and $y_i^0 > 0$ if and only if $v_i > \mu/a_i + \lambda/b_i$.

Lemma 7 characterises the solution, as it demonstrates that the criterion for attacking or defending a target depends on the quantity $\mu/a_i + \lambda/b_i$.

IV. DETERMINATION OF μ AND λ

If the condition

$$\sum_{i=1}^{k} x_i^0 = X$$

is written as

$$\sum_{y_i^0=0} x_i^0 + \sum_{y_i^0>0} x_i^0 = X,$$

then from Lemmas 4 and 5,

$$\sum_{\lambda/b_i < v_i \leq \mu/a_i + \lambda/b_i} (1/b_i) \ln (v_i b_i/\lambda) + \sum_{v_i > \mu/a_i + \lambda/b_i} (1/b_i) \ln [(\mu/a_i + \lambda/b_i)/(\lambda/b_i)] = X. \tag{6}$$

Similarly,

$$\sum_{v_i > \mu/a_i + \lambda/b_i} (1/a_i) \ln (v_i/(\mu/a_i + \lambda/b_i)) = Y. \tag{7}$$

The values of λ and μ are uniquely determined by (6) and (7) and hence the solutions for both the attacker and defender are unique. The fundamental theorem states that the game has a value V such that

$$V = \max_{\mathbf{x}} \min_{\mathbf{y}} F(\mathbf{x}, \mathbf{y}) = \min_{\mathbf{y}} \max_{\mathbf{x}} F(\mathbf{x}, \mathbf{y}),$$

and in this case

$$V = F(\mathbf{x}^0, \mathbf{y}^0) = \sum_{i=1}^{k} f_i(x_i^0, y_i^0).$$

By combining the results of Lemmas 4 and 5, this value may be written as

$$V = \lambda \sum_{i \in S_1} (1/b_i) + \sum_{i \in S_2} (v_i - \lambda/b_i),$$

where S_1, S_2 are the sets

$$S_1 = \{i : x_i^0 > 0 \text{ and } y_i^0 > 0\}, \qquad S_2 = \{i : x_i^0 > 0 \text{ and } y_i^0 = 0\}.$$

V. NUMERICAL EXAMPLE

In the special case where the vulnerability factors a_i and b_i are equal for each i, (6) and (7) reduce to

$$\sum_{\lambda < a_i v_i \leq \lambda + \mu} (1/a_i) \ln (a_i v_i/\lambda) + \sum_{a_i v_i > \lambda + \mu} (1/a_i) \ln [(\lambda + \mu)/\lambda] = X, \tag{9}$$

and

$$\sum_{a_i v_i > \lambda + \mu} (1/a_i) \ln [a_i v_i/(\lambda + \mu)] = Y. \tag{10}$$

The value of $(\lambda+\mu)$ may be obtained from (10) by systematically examining the possible intervals in which it can lie. When this value of $(\lambda+\mu)$ is substituted into (9), this equation involves only λ, and hence both λ and μ are determined. A similar procedure may be used for the case of unequal a_i and b_i, although in this case the calculations are more cumbersome. The method of the former case is illustrated in the following example of three $(k=3)$ targets with parameter values:

$$X=28, \quad Y=16, \quad a_1=0.03, \quad a_2=0.20, \quad a_3=0.20, \quad v_1=70, \quad v_2=80, \quad v_3=100.$$

From (10), the value of $(\lambda+\mu)$ is found from

$$\sum_{a_i v_i > \lambda+\mu} (1/a_i) \ln [a_i v_i/(\lambda+\mu)] = 16.$$

Suppose $24 < \lambda+\mu < 50$. Then

$$5 \ln [50/(\lambda+\mu)] = 16$$

which implies $(\lambda+\mu)=5.46$, which is a contradiction.

Suppose $0 < \lambda+\mu < 10$. Then

$$5 \ln [10/(\lambda+\mu)] + (10/3) \ln [24/(\lambda+\mu)] + 51 \ln [50/(\lambda+\mu)] = 16,$$

which implies $(\lambda+\mu)=7.78$ which does lie between 0 and 10.

Since $\mu > 0$ then $0 < \lambda < 10$. From (9),

$$\sum_{a_i v_i > 7.78} (1/a_i) \ln (7.78/\lambda) = 28$$

which yields $\lambda=3.86$. Therefore $\mu=3.92$.

The optimal strategies are easily found from Lemmas 4, 5, and 7 and the solution is given in Table 1.

TABLE 1

i	a_i	v_i	x_i^0	y_i^0
1	0.20	70	8.00	2.94
2	0.30	80	12.00	3.76
3	0.20	100	8.00	9.30

From (8) the value of the game is

$$V=\mu \sum_{i=1}^{3} 1/a_i = 52.20.$$

VI. REMARKS

The difficulty in finding the solution to games such as the one considered in this paper relies heavily

on the particular payoff function. At this stage in time there is no single method of solving resource allocation problems with a general payoff function, and extensive research is still being undertaken in this area.

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OFFICE OF NAVAL RESEARCH
Arlington, Va. 22217